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ON THE SOLUTIONS OF COUPLED SYSTEMS OF DELAY DIFFERENTIAL EQUATIONS

Ajaeb M. Abdulhamed
Faculty of science, University of Derna, Libya
ajaebelfadelelmansory@gmail.com
Samah S. Abushammala
Faculty of Economics, Omer Al-Mukhtar University, Libya
samah.said@omu.edu.ly

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Abstract

This paper studies the qualitative behavior of solutions to coupled systems of delay differential equations. Sufficient conditions for the existence and uniqueness of solutions are established using fixed point techniques in appropriate Banach spaces. examines the effect of coupling and time-delay parameters on the stability properties of the system. The results contribute to a deeper theoretical understanding of coupled delay differential systems and support their applicability in various mathematical and applied modeling contexts.

Keyword Coupled system, Delay differential equation, Fixed point theorem

1. Introduction

Delay differential equations (DDE) are differential equations in which there is time lag. This corresponds to a amount of time between a signal and response, providing a system feedback timescale.

Models of this form arise in applications biology, engineering, ecology, chemistry, and other systems containing derivatives which depend on a previous states [19].

Preliminaries and Notations

Through this thesis we will generally use the following notations:

(1)- Let $C = C(\mathrm{I})$ denotes the class of continuous functions defined on the interval $I = [0, T]$ with the norm

$$\|f\| = \sup \mathrm{e}^{-\mathrm{Nt}}{}_{\mathrm{t}\in[0,\mathrm{T}]} \; |f(t)\,)|, \qquad N > 0.$$

which is equivalent to the usual norm

$$||f|| = sup_{t\in(0,T)}|f(t)|$$

(2)- X denotes the class of all column vectors $\binom{x}{y}$ such that $x, y \in C(I)$

With the norm

$$\left\|\binom{x}{y}\right\| = \|x\| + \|y\|$$

(3)- Y denotes the class of continuous functions (x, y) such that $x, y \in C(I)$with the norm

$$\|(x, y)\| = \|x\| + \|y\|$$

Definition [13]

Assume that $f: I \times R \to R$ satisfies Caratheodory conditions, that is measurable in t for any x and continuous in x on the interval I ,we assigned the function

$$(Fx)(t) = f\big(t, x(t)\big), \ \ t \in I \ .$$

The operator F, defined in this way is called the superposition operator generated by the fu nction f.

Theorem [2]

The function $f(x) = f_1(x), f(x)_2, \ldots .. f_n(x))$ is uniformly continuous In $I = [a, b]$if and on ly if each f_i is uniformly continuous in $[a, b]$.

Theorem (Banach contraction mapping principle) [10]

Let X be a complete metric space and let $T: X \to X$ be a contraction map. Then T has a u nique fixed point in X .Moreover,for any $x_0 \in X$,the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ converge s to the fixed point.

This theorem is the most useful fixed point theorem, which is involved in many of the ex istence and uniqueness proofs in ordinary differential equations. The mapping T is the B anach contraction mapping principle stilI has a unique fixed point in any closed subset M of X. There are some conditions for a continuous mapping T in X.

Theorem (Schauder) [10]

Let Q be a convex subset of a Banach space X,and $T: Q \to Q$ is compact ,continuous map. Th en T has at least one fixed point in Q.

Definition [12]

let $F = f_i: X \to Y, i = I$ be a family of functions with y being a set of real (or complex) num bers, then we call F uniformly bounded if there exists a real number c such that $|f_i(x)| \le c$ $, \forall i \in I, x \in X$.

Definition [12]

Let $F = \{f(x)\}$ is the class of functions defined on $A = [a, b] \subset \mathcal{R}$ the class of functions $F = \{f(x)\}$ is equicontinuous if $\forall \epsilon > 0,\ \exists\ \delta(\epsilon)$ such that

$$|x - y| < \ \delta \Longrightarrow |f(x) - f(y)| < \epsilon \, , \forall f \in F \ and \ x, y \in A$$

Theorem (Arzela-Ascoli theorem)[12]

Let E be a compact metric space and $C(E)$ be the Banach space of real or complex valued c ontinuous functions norms by

$$||f|| = {}^{sup}_{t \in E} |f(t)|$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that fn is uniformly bounded and equi-continuous ,th en $\bar{A}$ is compact.

Theorem (Lebesgue dominated convergence theorem)[12]

Let $\{f_n\}$ be a sequence of functions convergingto a limit f on A, and suppose that

$|f_n(t)| \le \emptyset(t),\ \ t \in A.\ n = 1,2, \ldots ....$ where $\emptyset$ is integrable on A, then f is integrable on A and

$$\lim_{n\to\infty}\int_{A} f_n(t)d\mu = \int_{A} f(t)d\mu$$

2. Existence of a unique solution

The existence of a unique continuous solution $\binom{x}{y}$ for the coupled system of the delay differential equations.

$$\frac{dx}{dt} = f_1\big(t, y(t), y(t-r_1)\big) \qquad r_1, t > 0 \qquad (1)$$

$$\frac{dy}{dt} = f_2\big(t, x(t), x(t-r_2)\big) \qquad r_2, t > 0 \qquad (2)$$

subject to the data

$$x(t) = x_0 \qquad t \le 0 \quad (3)$$

$$y(t) = y_0 \qquad t \le 0 \quad (4)$$

Let X be the class of continuous column vectors with the norm $\binom{x}{y}$ such that $x, y \in C[0,T]$ *with the norm*

$$\left\|\binom{x}{y}\right\| = \|x\| + \|y\| = \sup_{t\in[o,T]}|x(t)| + \sup_{t\in[o,T]}|y(t)|$$

Consider the problem (1.1)- (1.4) under the following assumptions.

① $f_i: [0,T] \times R \times R \to R$ *are continuos*

② f_i *satisfy the Lipschitz condition*

$$|f_i(t,x_1,x_2) - f_i(t,y_1,y_2)| \le L_i(|x_1 - y_1| + |x_2 - y_2|)\,, L_i > 0,\ i = 1,2.$$

Now we have the following theorem

Theorem 1.2 Let the assumptions (1) and (2) are satisfied. If $2LT < 1$, where $L = \max(L_1, L_2)$ then the problem (1.1)- (1.4) has a unique solution $\binom{x}{y} \in X$.

Proof. The problem (1.1)- (1.4) can be written as

$$\frac{d}{dt}\binom{x}{y} = \begin{pmatrix} f_1\big(t, y(t), y(t-r_1)\big) \\ f_2\big(t, x(t), x(t-r_2)\big)\end{pmatrix}, \qquad \binom{x_0}{y_0} = X_0$$

Integrate both sides of the coupled system of the delay differential equations (1.1) and (1.2) , we obtain

$$\binom{x}{y} = X_0 + \begin{pmatrix} \int_0^{r_1} f_1(s, y(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y(s), y(s-r_1)\big)ds \\ \int_0^{r_2} f_2(s, x(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x(s), x(s-r_2)\big)ds \end{pmatrix} \qquad (9)$$

Define the operator F

$$F\binom{x}{y}=\binom{F_1x}{F_2y}=\begin{pmatrix} x_0+\int_0^{r_1} f_1(s,y(s),y_0)ds+\int_{r_1}^{t} f_1\big(s,y(s),y(s-r_1)\big)ds \\ y_0+\int_0^{r_2} f_2(s,x(s),x_0)ds+\int_{r_2}^{t} f_2\big(s,x(s),x(s-r_2)\big)ds \end{pmatrix}$$

Now

let $x\in C[0,T]$, then

$$|F_1x(t_2)-F_1x(t_1)|=$$

$$= \Big|x_0+\int_0^{r_1} f_1(s,y(s),y_0)\,ds+\int_{r_1}^{t_2} f_1\big(s,y(s),y(s-r_1)\big)\,ds \;-x_0- \int_0^{r_1} f_1(s,y(s),y_0)\,ds-\int_{r_1}^{t_1} f_1\big(s,y(s),y(s-\ r_1)\big)\,ds\Big|$$

$$= \Big|\int_{r_1}^{t_1} f_1\big(s,y(s),y(s-r_1)\big)\,ds+\int_{r_1}^{t_2} f_1\big(s,y(s),y(s-r_1)\big)\,ds- \int_{r_1}^{t_1} f_1\big(s,y(s),y(s-r_1)\big)\,ds\Big|$$

$$\le \int_{t_1}^{t_2}\big|f_1\big(s,y(s),y(s-r_1)\big)\big|\,ds$$

This implies that $F_1x\in C[o,T]$

Also, for $y\in C[0,T]$ then

$$|F_2y(t_2)-F_2y(t_1)|=\Big|y_0+\int_0^{r_2} f_2(s,x(s),x_0)\,ds+\int_{r_2}^{t_2} f_2\big(s,x(s),x(s-r_2)\big)\,ds-y_0-\int_0^{r_2} f_2(s,x(s),x_0)\,ds-\int_{r_2}^{t_1} f_2\big(s,x(s),x(s-r_2)\big)\,ds\Big|$$

$$=\Big|\int_{r_2}^{t_1} f_2\big(s,x(s),x(s-r_2)\big)\,ds+\int_{r_2}^{t_2} f_2\big(s,x(s),x(s-r_2)\big)\,ds- \int_{r_1}^{t_1} f_2\big(s,x(s),x(s-r_2)\big)\,ds\Big|\le\int_{t_1}^{t_2}\big|f_2\big(s,x(s),x(s-r_2)\big)\big|\,ds$$

This implies that $F_2y\in C[0,T]$

Hence $F\colon X\to X$

Let $U=\binom{x_1}{y_1}$ *and* $V=\binom{x_2}{y_2}$

Then

$$FU=F\binom{x_1}{y_1}=X_0+\begin{pmatrix} \int_0^{r_1} f_1(s,y_1(s),y_0)ds+\int_{r_1}^{t} f_1\big(s,y_1(s),y_1(s-r_1)\big)ds \\ \int_0^{r_2} f_2(s,x_1(s),x_0)ds+\int_{r_2}^{t} f_2\big(s,x_1(s),x_1(s-r_2)\big)ds \end{pmatrix}$$

and

$$FV = F\binom{x_2}{y_2} = X_0 + \begin{pmatrix} \int_0^{r_1} f_1(s, y_2(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y_2(s), y_2(s-r_1)\big)ds \\ \int_0^{r_2} f_2(s, x_2(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x_2(s), x_2(s-r_2)\big)ds \end{pmatrix}$$

Then

$FU - FV =$

$$\begin{pmatrix} \int_0^{r_1} f_1(s, y_1(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y_1(s), y_1(s-r_1)\big)ds \\ \int_0^{r_2} f_2(s, x_1(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x_1(s), x_1(s-r_2)\big)ds \end{pmatrix} - \begin{pmatrix} \int_0^{r_1} f_1(s, y_2(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y_2(s), y_2(s-r_1)\big)ds \\ \int_0^{r_2} f_2(s, x_2(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x_2(s), x_2(s-r_2)\big)ds \end{pmatrix}$$

And

$$\|FU - FV\| = \left\| \int_0^{r_1} f_1(s, y_1(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y_1(s), y_1(s-r_1)\big)ds - \int_0^{r_1} f_1(s, y_2(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y_2(s), y_2(s-r_1)\big)ds \right\| + \left\| \int_0^{r_2} f_2(s, x_1(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x_1(s), x_1(s-r_2)\big)ds - \int_0^{r_2} f_2(s, x_2(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x_2(s), x_2(s-r_2)\big)ds \right\|$$

But

$$\left| \int_0^{r_1} f_1(s, y_1(s), y_0)ds - \int_0^{r_1} f_1(s, y_2(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y_1(s), y_1(s-r_1)\big)ds - \int_{r_1}^{t} f_1\big(s, y_2(s), y_2(s-r_1)\big)ds \right|$$

$$\leq \left| \int_0^{r_1} f_1(s, y_1(s), y_0)ds - f_1(s, y_2(s), y_0)ds \right| + \left| \int_{r_1}^{t} f_1\big(s, y_1(s), y_1(s-r_1)\big)ds - f_1\big(s, y_2(s), y_2(s-r_1)\big)ds \right|$$

$$\leq \int_0^{\mathrm{r}_1} |f_1(s, y_1(s), y_0)ds - f_1(s, y_2(s), y_0)ds| + \int_{\mathrm{r}_1}^{\mathrm{t}} \left| f_1\big(s, y_1(s), y_1(s-r_1)\big)ds - f_1\big(s, y_2(s), y_2(s-r_1)\big)ds \right|$$

$$\leq \mathrm{L}_1 \int_0^{\mathrm{r}_1} |y_1(\mathrm{s}) - y_2(\mathrm{s})| + \mathrm{L}_1 \int_{\mathrm{r}_1}^{\mathrm{t}} |y_1(\mathrm{s}) - y_2(\mathrm{s})| + \mathrm{L}_1 \int_0^{\mathrm{t}} |y_1(\mathrm{s}) - y_2(\mathrm{s})| ds$$

$$\leq \mathrm{L}_1 \|y_1 - y_2\| \int_0^{\mathrm{r}_1} \mathrm{ds} + \mathrm{L}_1 \|y_1 - y_2\| \int_{\mathrm{r}_1}^{\mathrm{t}} \mathrm{ds} + \mathrm{L}_1 \|y_1 - y_2\| \int_{\mathrm{o}}^{\mathrm{t}} \mathrm{ds} \leq \mathrm{L}_1 \|y_1 - y_2\| (\mathrm{r}_1 + \mathrm{T} - \mathrm{r}_1 + \mathrm{T})$$

$\leq 2\mathrm{L}_1\mathrm{T}\|y_1 - y_2\|$.

And

$\left|\int_0^{r_1} f_2(s, x_1(s), x_0)ds - \int_0^{r_1} f_2(s, x_2(s), x_0)ds + \int_{r_1}^{t} f_2\big(s, x_1(s), x_1(s - r_1)\big)ds - \int_{r_1}^{t} f_2\big(s, x_2(s), x_2(s - r_1)\big)ds\right| \leq \left|\int_0^{r_1} f_2(s, x_1(s), x_0)ds - f_2(s, x_2(s), x_0)ds\right| + \left|\int_{r_1}^{t} f_2\big(s, x_1(s), x_1(s - r_1)\big)ds - f_2\big(s, x_2(s), x_2(s - r_1)\big)ds\right| \leq \int_0^{\mathrm{r}_1}|f_2(s, x_1(s), x_0)ds - f_2(s, x_2(s), x_0)ds| + \int_{\mathrm{r}_1}^{\mathrm{t}}\big|f_2\big(s, x_1(s), x_1(s - r_1)\big)ds - f_2\big(s, x_2(s), x_2(s - r_1)\big)ds\big| \leq \mathrm{L}_1 \int_0^{\mathrm{r}_1}|x_1(\mathrm{s}) - x_2(\mathrm{s})| + \mathrm{L}_1 \int_{\mathrm{r}_1}^{\mathrm{t}}|x_1(\mathrm{s}) - x_2(\mathrm{s})| + \mathrm{L}_1 \int_0^{\mathrm{t}}|x_1(\mathrm{s}) - \mathrm{x}x_2(\mathrm{s})|ds \leq \mathrm{L}_1\|x_1 - x_2\| \int_0^{\mathrm{r}_1} \mathrm{ds} + \mathrm{L}_1\|x_1 - x_2\| \int_{\mathrm{r}_1}^{\mathrm{t}} \mathrm{ds} + \mathrm{L}_1\|x_1 - x_2\| \int_{\mathrm{o}}^{\mathrm{t}} \mathrm{ds} \leq \mathrm{L}_1\|x_1 - x_2\|(\mathrm{r}_1 + \mathrm{T} - \mathrm{r}_1 + \mathrm{T}) \leq 2\mathrm{L}_1\mathrm{T}\|x_1 - x_2\|$

Then

$$\|\mathrm{FU} - \mathrm{FV}\| \leq 2L_1T\|y_1 - y_2\| + 2L_2T\|x_1 - x_2\|$$
$$\leq 2LT(\|x_1 - x_2\| + \|y_1 - y_2\|)$$
$$\leq 2LT\|U - V\|.$$

*W*here $L = max(L_1, L_2)$. If $2LT < 1, then\ F\ is\ contraction.$

Using the Banach fixed point theorem we deduce that there exists a unique solution $\binom{x}{y} \in X, of\ the\ integral\ equation$ (9).

To complete the proof, differential both sides (9), we obtain the delay differenti al equations (1)-(2)

Letting $t \leq 0\ in$ (9), We obtain the initial data (3)-(4).

3. Existence of at least one solution

We study the existence of at least one continuous solution for the coupled system of the delay differential equations (1.1)-(1.2).

Finally, we study the existence of a unique uniformly stable solution $\binom{x}{y}$ the coupled system of the delay differential equations for

$$\frac{dx}{dt} = f_1\big(t, y(t), y(t - r)\big), \ t > r > 0 \qquad (5)$$

$$\frac{dy}{dt} = f_2\big(t, x(t), x(t - r)\big), \ t > r > 0 \qquad (6)$$

subject to the data

$x(t) = x_0 \ , \ t \leq r \qquad (7)$

$y(t) = y_0 \ , \ t \leq r \qquad (8)$

Let Y be the class of continuous functions (x, y) , $x, y \in C[0, T]$, with the norm

$$\|(x, y)\| + \|x\| + \|y\| = \sup_{t\in[o,T]}|x(t)| + \sup_{t\in[o,T]}|y(t)|$$

Consider the problem (1) - (2) under the following assumptions

(1) $f_{\mathrm{i}}: [\mathrm{o}, \mathrm{T}] \times \mathrm{R} \times \mathrm{R}$ Satisfy Caratheodory condition ,that is are measurable in $t \in [o, T]$ *for any* $x \in R$ *and continous in* $x \in R$ *for almost all* $t \in [0, T]$

(2) *there exist integrable functions* $m_i \in L^1[0, T]$ *, and positive constants* $b_1, b_2, b_3, b_4 > 0$ *such that*

$|f_1(\mathrm{t}, x_1, x_2)| \leq m_1 + b_1|x_1| + \mathrm{b}_2|x_2|$, $|f_2(t, y_1, y_2)| \leq m_2 + b_2|y_1| + b_2|y_2|$

(3) $\int_0^t m_i(s)ds < k_i$, $i = 1,2$.

Now we have the following theorem

Theorem Assume that (1*),(2*) and (3) are satisfied, then there exists at least one solution coupled system of delay differential equations (1) (4).

proof. Integral both sides of the coupled system of delay differential equations (1) -(2), we obtain the integral equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \int_0^{r_1} f_1(s, y(s), y_0)ds + \int_{r_1}^t f_1\big(s, y(s), y(s - r_1)\big)ds \\ \int_0^{r_2} f_2(s, x(s), x_0)ds + \int_{r_2}^t f_2\big(s, x(s), x(s - r_2)\big)ds \end{pmatrix}$$

Define the operator F by

$$F(x, y) = (F_1x, F_2x) = \Big(x_0 + \int_0^{r_1} f_1(s, y(s), y_0)ds + \int_{r_1}^t f_1\big(s, y(s), y(s - r_1)\big)ds \, , y_0 + \int_0^{r_2} f_2(s, x(s), x_0)ds + \int_{r_2}^t f_2\big(s, x(s), x(s - r_2)\big)ds\Big)$$

Define the set $Q = \{(x, y) \in Y : \|(x, y)\| + \|x\| + \|y\| < M\}$.

Now

$$|F_1x| = \left|x_0 + \int_0^{r_1} f_1(s, y(s), y_0)ds + \int_{r_1}^t f_1\big(s, y(s), y(s - r_1)\big)ds\right|$$

$$\leq |x_0| + \int_0^{r_1}|f_1(s, y(s), y_0)|ds + \int_{r_1}^t\big|f_1\big(s, y(s), y(s - r_1)\big)\big|ds$$

$$\leq |x_0| + \int_0^{r_1} m_1(s)ds + b_1 \int_0^{r_1}|y(s)|ds + b_2 \int_0^{r_1}|y_0|ds + \int_{r_1}^t m_1(s)\, ds + b_1 \int_{r_1}^t|y(s)|ds + b_2 \int_0^t|y(s)|ds$$

$$\leq |x_0| + 2k_1 + 3B_1\|y\|T + b_2|y_0|T = N_1$$

Where $B_1 = max(b_1, b_2)$, Hence F_1 is uniformly bounded.

Similarly we have

$$|F_2y| = \left|y_0 + \int_0^{r_2} f_2(s, x(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x(s), x(s - r_2)\big)ds\right|$$

$$\leq |y_0| + \int_0^{r_2}|f_2(s, x(s), x_0)|ds + \int_{r_2}^{t}\left|f_2\big(s, x(s), x(s - r_2)\big)\right|ds$$

$$\leq |y_0| + \int_0^{r_2} m_1(s)ds + b_3 \int_0^{r_1}|(s)|ds + b_4 \int_0^{r_1}|x_0|ds$$

$$+ \int_{r_1}^{t} m_2(s)\, ds + b_3 \int_{r_1}^{t}|x(s)|ds + b_4 \int_0^{t}|x(s)|ds$$

$$\leq |y_0| + 2k_2 + 3B_2\|x\|T + b_4|x_0|T = N_2 \quad .$$

Where $B_2 = max(b_3, b_4)$. Hence F_2 is uniformly bounded.

Combining the results we obtain

$$\|F(x, y)\| = \|F_1x\| + \|F_2y\| \leq N_1 + N_2 \leq M$$

Which implies that F is uniformly bounded.

We have For $t_2, t_1 > r_1,\ \ t_2 > t_1 \ and \ |t_2 - t_1| < \delta,$

$$|F_1x(t_2) - F_1x(t_1)| = \left|x_0 + \int_{r_1}^{t_2} f_1\big(s, y(s), y(s - r_1)\big)ds - x_0 - \int_{r_1}^{t_1} f_1\big(s, y(s), y(s - r_1)\big)ds\right|$$

$$= \left|\int_{r_1}^{t_1} f_1\big(s, y(s), y(s - r_1)\big)ds + \int_{r_1}^{t_2} f_1\big(s, y(s), y(s - r_1)\big)ds - \int_{r_1}^{t_1} f_1\big(s, y(s), y(s - r_1)\big)ds\right|$$

$$\leq \int_{r_1}^{t_2}\left|\int_{r_1}^{t_2} f_1\big(s, y(s), y(s - r_1)\big)ds\right|$$

$$\leq \int_{t_1}^{t_2} m_1(s)ds + b_1 \int_{t_1}^{t_2}|y(s)|ds + b_2 \int_{t_1-r_1}^{t_2-r_1}|y(s)|ds$$

$$\leq \int_{t_1}^{t_2} m_1(s)ds + 2M(t_2 - t_1).$$

Hence $\{F_1x\}$ is class of equicontinuous.

For $t_2, t_1 > r_2\ , t_2 > t_1$ and $|t_2 - t_1| < \delta$ we have

$$|F_2x(t_2) - F_2x(t_1)| = \left|y_0 + \int_{r_2}^{t_2} f_2\big(s, x(s), x(s - r_2)\big)ds - y_0 - \int_{r_2}^{t_1} f_2\big(s, x(s), x(s - r_2)\big)ds\right|$$

$$= \left|\int_{r_2}^{t_1} f_2\big(s, x(s), x(s - r_2)\big)ds + \int_{r_1}^{t_2} f_2\big(s, x(s), x(s - r_2)\big)ds - \int_{r_2}^{t_1} f_2\big(s, x(s), x(s - r_2)\big)ds\right|$$

$$\leq \int_{r_1}^{t_2}\left|\int_{r_1}^{t_2} f_2\big(s, x(s), x(s - r_2)\big)ds\right|$$

$$\leq \int_{t_1}^{t_2} m_2(s)ds + b_3 \int_{t_1}^{t_2}|x(s)|ds + b_4 \int_{t_1-r_2}^{t_2-r_2}|x(s)|ds$$

$$\leq \int_{t_1}^{t_2} m_2(s)ds + 2M(t_2 - t_1).$$

Hence $\{F_2 y\}$ is class of equicontinuous.

This implies that F is equicontinuous function.

Therefor the operator F is equicontinuous and uniformly bounded, from Arzela – Ascoli theorem F is compact.

Now

Let $\{y_n\}$ be convergent sequence such that $y_n \to y$,

Then

$$\lim_{n\to\infty} F_1 x_n = x_0 + \lim_{n\to\infty} \int_0^{r_0} f_1(s, y(s), y_0) ds + \lim_{n\to\infty} \int_{r_1}^{t} f_1\big(s, y(s), y(s-r_1)\big) ds$$

But from the assumption

$$|f_1(t, y_n, y_n)| \leq m_1 + b_1|y_n| + b_2|y_n|$$

$$\leq m_1 + \mathrm{b}_1 M + b_2 M$$

And $f_1(t, y_n, y_n) \to f_1(t, y, y)$

Applying Lebesgue dominated convergence theorem , then

$$\lim_{n\to\infty} F_1 x_n = x_0 + \lim_{n\to\infty} \int_0^{r_1} f_1(s, y(s), y_0) ds + \lim_{n\to\infty} \int_{r_1}^{t} f_1\big(s, y(s), y(s-r_1)\big) ds = F_1 x$$

Which proves that F_1 is continuous operator.

Let $\{x_n\}$ be convergent sequence such that $x_n \to x$, then

$$\lim_{n\to\infty} F_2 y_n = y_0 + \lim_{n\to\infty} \int_0^{r_2} f_2(s, x(s), x_0) ds + \lim_{n\to\infty} \int_{r_2}^{t} f_2\big(s, x(s), x(s-r_2)\big) ds$$

But from the assumption

$$|f_2(t, x_n, x_n)| \leq m_2 + b_3|x_n| + b_4|x_n|$$

$$\leq m_2 + \mathrm{b}_3 M + b_4 M$$

And $f_2(t, x_n, x_n) \to f_2(t, x, x)$

Applying Lebesgue dominated convergence theorem , then

$$\lim_{n\to\infty} F_2 y_n = y_0 + \lim_{n\to\infty} \int_0^{r_2} f_2(s, x(s), x_0) ds + \lim_{n\to\infty} \int_{r_2}^{t} f_2\big(s, x(s), x(s-r_2)\big) ds = F_2 y$$

Which proves that F_2 is continuous operator.

Hence $F: Q \to Q$ is continuous and compact using Schauder fixed point theorem has a fixed point $\binom{x}{y} \in X$ which proves that there exists at least one solution (9) of the coupled system of the delayed fifferential equations (1)- (2).

$$\binom{x}{y} = \binom{x_0}{y_0} + \begin{pmatrix} \int_0^{r_1} f_1(s, y(s), y_0)ds + \int_{r_1}^{t} f_1\big(s, y(s), y(s - r_1)\big)ds \\ \int_0^{r_2} f_2(s, x(s), x_0)ds + \int_{r_2}^{t} f_2\big(s, x(s), x(s - r_2)\big)ds \end{pmatrix}$$

Complete the proof , differential both sides we obtain the delay differential equations (1)-(2).

Letting in (9), we obtain the initial data (3)-(4).

4. Stability of solution

Here, We study the uniform stability of the solution of the problem (5) - (8) Consider the two coupled system of the delay differential equations (5) - (6)

subject to the data

$$x(t) = \bar{x}_0, \quad t \le r \qquad (10)$$

$$y(t) = \bar{y}_0, \quad t \le r \qquad (11)$$

Definition The system of differential equations (5)- (6) is uniformly stable if $\forall\, \epsilon > 0,\ \exists\, \delta > 0,$ such that

$$|x_0 - \bar{x}_0| < \frac{\delta}{2} \quad and \quad |y_0 - \bar{y}_0| < \frac{\delta}{2} \ \Rightarrow \left\| \binom{x}{y} - \binom{\bar{x}}{\bar{y}} \right\| < \epsilon$$

Now we have the following theorem

Theorem Let the assumptions of the Theorem (4) are satisfied, then the solution of the coupled system (5) - (6) is uniformly stable.

Proof. The solutions of the differential equation (5), (8) and (5), (6),(10), (11) are given by

$$U(t) = \binom{x(t)}{y(t)} = \begin{pmatrix} x_0 + \int_0^{r} f_1(s, y(s), y_0)ds + \int_{r}^{t} f_1\big(s, y(s), y(s - r)\big)ds \\ y_0 + \int_0^{r} f_2(s, x(s), x_0)ds + \int_{r}^{t} f_2\big(s, x(s), x(s - r)\big)ds \end{pmatrix}$$

And

$$\bar{U}(t) = \binom{\bar{x}(t)}{\bar{y}(t)} = \begin{pmatrix} \bar{x}_0 + \int_0^{r} f_1(s, \bar{y}(s), \bar{y}_0)ds + \int_{r}^{t} f_1\big(s, \bar{y}(s), \bar{y}(s - r)\big)ds \\ \bar{y}_0 + \int_0^{r} f_2(s, \bar{x}(s), \bar{x}_0)ds + \int_{r}^{t} f_2\big(s, \bar{x}(s), \bar{x}(s - r)\big)ds \end{pmatrix}$$

Let $|x_0 - \bar{x}_0| < \frac{\delta}{2} \quad and \quad |y_0 - \bar{y}_0| < \frac{\delta}{2},$

Then

$U - \bar{U} =$

$$\begin{pmatrix} x_0 + \int_0^r f_1(s,y(s),y_0)ds + \int_r^t f_1\big(s,y(s),y(s-r)\big)ds \\ y_0 + \int_0^r f_2(s,x(s),x_0)ds + \int_r^t f_2\big(s,x(s),x(s-r)\big)ds \end{pmatrix} - \begin{pmatrix} \bar{x}_0 + \int_0^r f_1(s,\bar{y}(s),\bar{y}_0)ds + \int_r^t f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)ds \\ \bar{y}_0 + \int_0^r f_2(s,\bar{x}(s),\bar{x}_0)ds + \int_r^t f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)ds \end{pmatrix}$$

And

$\|U-\bar{U}\| \le \Big\| x_0 - \bar{x}_0 + \int_0^r f_1(s,y(s),y_0)ds - \int_0^r f_1(s,\bar{y}(s),\bar{y}_0)ds + \int_r^t f_1\big(s,y(s),y(s-r)\big)ds - \int_r^t f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)ds \Big\|$

$+\Big\| y_0 - \bar{y}_0 + \int_0^r f_2(s,x(s),x_0)ds - \int_0^r f_2(s,\bar{x}(s),\bar{x}_0)ds + \int_r^t f_2\big(s,x(s),x(s-r)\big)ds - \int_r^t f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)ds \Big\|$

$\le |x_0-\bar{x}_0| + |y_0-\bar{y}_0| + \Big\| \int_0^r f_1(s,y(s),y_0)ds - \int_0^r f_1(s,\bar{y}(s),\bar{y}_0)ds + \int_r^t f_1\big(s,y(s),y(s-r)\big)ds - \int_r^t f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)ds \Big\| + \Big\| \int_0^r f_2(s,x(s),x_0)ds - \int_0^r f_2(s,\bar{x}(s),\bar{x}_0)ds + \int_r^t f_2\big(s,x(s),x(s-r)\big)ds - \int_r^t f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)ds \Big\|.$

But

$\Big| \int_0^r f_1(s,y(s),y_0)ds - \int_0^r f_1(s,\bar{y}(s),\bar{y}_0)ds + \int_r^t f_1\big(s,y(s),y(s-r)\big)ds - \int_r^t f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)ds \Big|$

$\le \Big| \int_0^r f_1(s,y(s),y_0)ds - \int_0^r f_1(s,\bar{y}(s),\bar{y}_0)ds + \int_r^t f_1\big(s,y(s),y(s-r)\big)ds - \int_r^t f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)ds \Big|$

$\le \int_0^r |f_1(s,y(s),y_0) - f_1(s,\bar{y}(s),\bar{y}_0)|\, ds + \int_r^t \big|f_1\big(s,y(s),y(s-r)\big) - f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)\big| ds$

$\le L_1 \int_0^r |y(s)-\bar{y}(s)|\, ds + L_1 \int_0^r |y_0-\bar{y}_0|\, ds + L_1 \int_r^t |y(s)-\bar{y}(s)|\, ds + L_1 \int_r^t |y(s)-\bar{y}(s)|\, ds$

$\le L_1 \int_0^r |y(s)-\bar{y}(s)|\, ds + L_1 \int_0^r |y_0-\bar{y}_0|\, ds + 2L_1 \int_r^t |y(s)-\bar{y}(s)|\, ds$

Then

$e^{-Nt} \Big| \int_0^r f_1(s,y(s),y_0)ds - \int_0^r f_1(s,\bar{y}(s),\bar{y}_0)ds + \int_r^t f_1\big(s,y(s),y(s-r)\big)ds - \Big| \int_r^t f_1\big(s,\bar{y}(s),\bar{y}(s-r)\big)ds$

$\le L_1 \int_0^r e^{-N(t-s)} e^{-Ns}\, |\mathrm{y(s)} - \bar{\mathrm{y}}(\mathrm{s})| ds + e^{-Nt} \mathrm{L}_1 |y_0-\bar{y}_0| \int_0^{\mathrm{r}} ds + 2L_1 \int_0^t e^{-N(t-s)}\, e^{-Ns} |\mathrm{y(s)} - \bar{\mathrm{y}}(\mathrm{s})| ds$

$\le L_1 \|y-\bar{y}\| \int_0^r e^{-N(t-s)} ds + e^{-Nt} \mathrm{L}_1 |y_0-\bar{y}_0| \int_0^{\mathrm{r}} ds + 2L_1\, \|y-\bar{y}\| \int_0^t e^{-N(t-s)}\, ds$

$\leq L_1\|y-\bar{y}\|\left(\left(\frac{e^{-N(t-r)}}{\mathrm{N}}-\frac{\mathrm{e}^{-\mathrm{Nt}}}{\mathrm{N}}\right)+e^{-Nt}\mathrm{L}_1|y_0-\bar{y}_0|r+\int_0^{\mathrm{r}} ds+2L_1\|y-\bar{y}\|\left(\frac{1}{\mathrm{N}}-\frac{\mathrm{e}^{-\mathrm{Nt}}}{\mathrm{N}}\right)\right)$

$\leq L_1\|y-\bar{y}\|\frac{e^{-N(t-r)}}{\mathrm{N}}+\mathrm{e}^{-\mathrm{Nt}}\mathrm{L}_1|y_0-\bar{y}_0|r+2\mathrm{L}_1\|\mathrm{y}-\bar{\mathrm{y}}\|\frac{1}{\mathrm{N}}$

$\leq \mathrm{e}^{-\mathrm{Nt}}\mathrm{L}_1|y_0-\bar{y}_0|r+\frac{L_1}{\mathrm{N}}\|y-\bar{y}\|+2\frac{L_1}{\mathrm{N}}\|y-\bar{y}\|$

$\leq \mathrm{e}^{-\mathrm{Nt}}\mathrm{L}_1|y_0-\bar{y}_0|r+3\frac{L_1}{\mathrm{N}}\|y-\bar{y}\|.$

Similarly we have

$\left|\int_0^r f_2(s,x(s),x_0)ds-\int_0^r f_2(s,\bar{x}(s),\bar{x}_0)ds+\int_r^t f_2\big(s,x(s),x(s-r)\big)ds-\int_r^t f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)ds\right|$

$\leq \left|\int_0^r f_2(s,x(s),x_0)ds-\int_0^r f_2(s,\bar{x}(s),\bar{x}_0)ds+\int_r^t f_2\big(s,x(s),x(s-r)\big)ds-\int_r^t f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)ds\right|$

$\leq \int_0^r|f_2(s,x(s),x_0)-f_2(s,\bar{x}(s),\bar{x}_0)|\,ds+\int_r^t\big|f_2\big(s,x(s),x(s-r)\big)-f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)\big|ds$

$\leq L_2\int_0^r|x(s)-\bar{x}(s)|\,ds+L_2\int_0^r|x_0-\bar{x}_0|\,ds+L_2\int_r^t|x(s)-\bar{x}(s)|\,ds+L_2\int_r^t|x(s)-\bar{x}(s)|\,ds$

$\leq L_2\int_0^r|x(s)-\bar{x}(s)|\,ds+L_2\int_0^r|x_0-\bar{x}_0|\,ds+2L_2\int_r^t|x(s)-\bar{x}(s)|\,ds$

Then

$e^{-Nt}\left|\int_0^r f_2(s,x(s),x_0)ds-\int_0^r f_2(s,\bar{x}(s),\bar{x}_0)ds+\int_r^t f_2\big(s,x(s),x(s-r)\big)ds-\right|\int_r^t f_2\big(s,\bar{x}(s),\bar{x}(s-r)\big)ds$

$\leq L_2\int_0^r e^{-N(t-s)}e^{-Ns}\,|x(\mathrm{s})-\bar{x}(\mathrm{s})|ds+e^{-Nt}L_2|x_0-\bar{x}_0|\int_0^{\mathrm{r}} ds+2\,L_2\int_0^t e^{-N(t-s)}\,e^{-Ns}|x(\mathrm{s})-\bar{x}(\mathrm{s})|ds$

$\leq L_2\|x-\bar{x}\|\int_0^r e^{-N(t-s)}ds+e^{-Nt}\mathrm{L}_2|x_0-\bar{x}_0|\int_0^{\mathrm{r}} ds+2L_2\|x-\bar{x}\|\int_0^t e^{-N(t-s)}\,ds$

$\leq L_2\|x-\bar{x}\|\left(\left(\frac{e^{-N(t-r)}}{\mathrm{N}}-\frac{\mathrm{e}^{-\mathrm{Nt}}}{\mathrm{N}}\right)+e^{-Nt}\mathrm{L}_2|x_0-\bar{x}_0|r+\int_0^{\mathrm{r}} ds+2L_2\|x-\bar{x}\|\left(\frac{1}{\mathrm{N}}-\frac{\mathrm{e}^{-\mathrm{Nt}}}{\mathrm{N}}\right)\right)$

$\leq L_2\|x-\bar{x}\|\frac{e^{-N(t-r)}}{\mathrm{N}}+\mathrm{e}^{-\mathrm{Nt}}L_2|x_0-\bar{x}_0|r+2\mathrm{L}_2\|x-\bar{x}\|\frac{1}{\mathrm{N}}$

$\leq \mathrm{e}^{-\mathrm{Nt}}L_2|x_0-\bar{x}_0|r+\frac{L_2}{\mathrm{N}}\|x-\bar{x}\|+2\frac{L_2}{\mathrm{N}}\|x-\bar{x}\|$

$\leq \mathrm{e}^{-\mathrm{Nt}}L_2|x_0-\bar{x}_0|r+3\frac{L_2}{\mathrm{N}}\|x-\bar{x}\|.$

Hence

$$\|U-\bar{U}\| \leq \delta + \mathrm{e}^{-\mathrm{Nt}}\mathrm{L}_1|y_0-\bar{y}_0|r + 3\frac{L_1}{\mathrm{N}}\|y-\bar{y}\| + \mathrm{e}^{-\mathrm{Nt}}L_2|x_0-\bar{x}_0|r + 3\frac{L_2}{\mathrm{N}}\|x-\bar{x}\|$$

$$\leq \delta + \mathrm{e}^{-\mathrm{Nt}}L_{\mathrm{r}}(|y_0-\bar{y}_0| + |x_0-\bar{x}_0|) + 3\frac{\mathrm{L}}{\mathrm{N}}(\|y-\bar{y}\| + \|x-\bar{x}\|)$$

$$\leq \left(\mathrm{e}^{-\mathrm{Nt}}L_{\mathrm{r}}+1\right)\delta + 3\frac{\mathrm{L}}{\mathrm{N}}\|U-\bar{U}\|$$

Where $L = max(L_1, L_2)$.

Then

$$\|U-\bar{U}\|\left(1-3\frac{L}{N}\right) \leq \left(\mathrm{e}^{-\mathrm{Nt}}L_{\mathrm{r}}+1\right)\delta$$

Hence

$$\|U-\bar{U}\| \leq \left(1-3\frac{L}{N}\right)^{-1}\left(\mathrm{e}^{-\mathrm{Nt}}L_{\mathrm{r}}+1\right)\delta = \epsilon. \quad \blacksquare$$

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