The Containment Relationships Between the Varieties of m-groups

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Abstract

The nation of an m-group as algebraic system is relatively new and was intrudes by (Giraudet, M. and Rachunek, J.,1999) when they were studying order properties of monotonic permutation groups of linearly ordered sets. In the present study the containment relationships between the varieties of m-groups will be exhibited where we will prove that the set of all varieties of m-groups are lattice and semigroup. The connections between the varieties of ξm, 𝓐m, 𝓡m, 𝓝m and 𝓐nm will be described.

Key words: order, variety, po-group, l-group, m-group.

الخلاصة

التنويعات لأي تركيبة جبرية هي صنف معرف بشكل تعادلي. أول من درس هذه الأصناف هو العالم Birkhoff سنة 1935 حيث أثبت أن أي صنف من التركيبات الجبرية يكون تنويع إذا كان مغلق تحت الضرب المباشر والتركيبة الجبرية الجزئية والصورة الهومومورفيه.

مفهوم m-goups كنظام جبري هو موضوع جديد نسبياً وقد تم طرحه بواسطة M. Giraudet و J. Rachunek [8]، عندما كانوا يدرسون خصائص الترتيب لمجموعات التبديل الرتيبة للمجموعات المرتبة خطياً. تهدف هذه الدراسة إلى مناقشة بعض الروابط وعلاقات الاحتواء بين التنويعات ل m-groups، حيث سنثبت أن مجموعة جميع أصناف m-groups تمثل شبكة وشبه مجموعة. سيتم وصف الروابط بين أصناف ξm, 𝓐m, 𝓡m, 𝓝m 𝓐nm ..

Section 1: Introduction

A partially ordered group (po-group) is a group ($\boldsymbol{G}$, +) with a partial order ≤ that is compatible with the group operation; that is, for all a, $b, x, y \in \boldsymbol{G}$

$$a \leq b \text{ implies } \quad x + a + y \leq x + b + y$$

If the partial order is a lattice order, then is called a lattice ordered group (ℓ-group). If the partial order is a total order, then $\boldsymbol{G}$ is called a totally ordered group (o-group) (*Fuchs, L. 1963, P9)* .

A variety of ℓ-groups is a non-empty collection of ℓ-groups closed under ℓ-subgroups, ℓ-homomorphic images, and cardinal direct products. Equivalently, it is the collection of all ℓ-groups defined by some (possibly infinite) set of equations (involving the group and lattice operations). The set of all ℓ-group varieties is a complete lattice. A variety $\mathcal{V}$ is said to be generated by {Gi | $i \in \boldsymbol{I}$} $\in \mathcal{V}$ if $\mathcal{V}$ is the smallest variety containing {Gi | $i \in \boldsymbol{I}$}

$$\ell\text{-}var\ \{G_i \mid i \in \boldsymbol{I}\} = \bigcap \{\mathcal{V} \in \boldsymbol{L} \mid \mathcal{V} \supseteq \{G_i \mid i \in \boldsymbol{I}\}\}.$$

The set of all ℓ-group varieties is a semigroup where $\boldsymbol{G}$ belongs to $\mathcal{UV}$ if and only if there exists an ℓ-ideal $\boldsymbol{H}$ of $\boldsymbol{G}$ such that $\boldsymbol{H} \in \mathcal{U}$ and $G/\boldsymbol{H} \in \mathcal{V}$.

Let $\boldsymbol{\mathcal{L}}$ be the variety of all ℓ-groups; $\boldsymbol{\mathcal{E}}$ the variety of all one-element ℓ-groups; $\boldsymbol{\mathcal{A}}$ the variety of all abelian ℓ-groups; $\boldsymbol{\mathcal{R}}$ the variety of all representable ℓ-groups (those ℓ-groups that are subdirect products of totally ordered groups); and $\boldsymbol{\mathcal{N}}$ the variety of all normal-valued ℓ-groups *(Reilly, N.1989,*P228).

An $\boldsymbol{m}$-*group* is an algebraic system $\boldsymbol{H}$ of signature $\boldsymbol{m} = (\boldsymbol{H}, e, ^{-1}, \vee, \wedge, \varphi)$, where $(\boldsymbol{H}, e, ^{-1}, \vee, \wedge)$ is an ℓ-group and the unary operation φ is an order 2 automorphism of the group $(\boldsymbol{H}, e, ^{-1})$ and an anti-automorphism of the lattice $(\boldsymbol{H}, \vee, \wedge,)$. (*Giraudet, M. and Rachunek, J. 1999,*P745).

Let $\boldsymbol{H}$ be ℓ-group, and for all $u \in \boldsymbol{H}$ we denote $\varphi_{\boldsymbol{u}}\boldsymbol{:}\ \boldsymbol{H} \to \boldsymbol{H}$ be the inner automorphism of $\boldsymbol{H}$ defined by $\varphi_{\boldsymbol{u}}(x) = uxu^{-1}$.

For two $\boldsymbol{m}$-group structures $(\boldsymbol{H}, \varphi)$ and $(\boldsymbol{H}, \varphi')$ in $\boldsymbol{I(H)}$, then we defined relation on $\boldsymbol{I(H)}$ by $(\boldsymbol{H}, \varphi)\ \#\ (\boldsymbol{H}, \varphi')$ if and only if for some $u \in \boldsymbol{H}$, $\varphi(u) = u^{-1}$ and $\varphi\varphi' = \varphi_{\boldsymbol{u}}$.

*Example*1.1: consider the group

$$\boldsymbol{G_2} = \{a_0, a_1, b \mid [a_0, a_1] = e, a_0^{\,b} = a_1, a_1^{\,b} = a_0\}.$$

Each $g \in \boldsymbol{G_2}$ has unique representation in the form $g = b^k a_0^{\,m} a_1^{\,n}$, $k, m, n \in \mathbb{Z}$. With respect to the lexicographic order

$$g \geq e \Leftrightarrow k = 0 \text{ or } k > 0,\ m \geq 0,\ n \geq 0$$

$\boldsymbol{G_2}$ is an ℓ-group. Define the mapping φ: $\boldsymbol{G_2} \to \boldsymbol{G_2}$ by

$$\varphi(g) = b^{-k} a_0^{\,-m} a_1^{\,-n};$$

then $(\boldsymbol{G_2}, \varphi)$ is $\boldsymbol{m}$-group.

*Example*1.2**.** Let $\boldsymbol{G}$ be a m-group and $\boldsymbol{G}\!\downarrow\ \neq \emptyset$ then there is an element $u \in \boldsymbol{G}\!\downarrow$ with $u^2 = e$, and if $\varphi_{\boldsymbol{u}}\boldsymbol{:}\ \boldsymbol{G}\!\uparrow \to \boldsymbol{G}\!\uparrow$ is the inner automorphism defined by $\varphi_{\boldsymbol{u}}(x) = uxu^{-1}$ then $(\boldsymbol{G}\uparrow, \varphi_{\boldsymbol{u}})$ is $\boldsymbol{m}$-group. *(Giraudet, M. and Lucas, F. 1991).*

Section 2: The ordered semigroup Of Varieties Of *m*-Groups.

The class of all ***m***-groups, denoted by $\mathscr{M}$, forms a variety of signature ***m*** .Let ***M*** be the set of all varieties of ***m***-groups**,** it is natural to defined a partial ordered on the set of all ***m***-groups varieties ***M***, by using set containment.

*For all $\mathcal{U}$ and $\mathcal{V}$ in **M**, define $\mathcal{V} \leq \mathcal{U}$ **if** and only if $\mathcal{V} \subseteq \mathcal{U}$.*

The partial order $\leq$ on ***M*** becomes a lattice order, if one defines for $\mathcal{V}_i \in \boldsymbol{M}$ $(i \in I)$

$$\bigwedge_{i\in I} \mathcal{V}_i = \bigcap_{i\in I} \mathcal{V}_i,$$

$$\bigvee_{i\in I} \mathcal{V}_I = \bigcap_{i\in I} \{\mathcal{U} \in \boldsymbol{M} \mid \mathcal{U} \geq \mathcal{V}_i,\ i \in I\}.$$

Since ***M*** contains both a greatest element $\boldsymbol{\mathcal{M}}$ is the largest variety of *m*-groups, since it contains every other variety of *m*-groups, and the least element is the trivial variety $\boldsymbol{\xi_m}$ where is the smallest variety of *m*-groups, since it is contained in every other variety of *m*-groups. The definition of $\bigwedge$ and $\bigvee$ on ***M*** suffice to make ***M*** a complete lattice.

Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of *m*-groups. Then the *product of $\mathcal{U}$ and $\mathcal{V}$* is the variety $\mathcal{U}\mathcal{V}$ defined by: $(\boldsymbol{G}, \varphi) \in \mathcal{U}\mathcal{V}$ if and only if there is an *m*-homomorphism of $(\boldsymbol{G}, \varphi)$ onto an element in $\mathcal{V}$ with the kernel in $\mathcal{U}$. (In other words: There is an *m*-ideal *A* of $(\boldsymbol{G}, \varphi)$ such that $A \in \mathcal{U}$ and $\boldsymbol{G}/A \in \mathcal{V}$). (*Zenkov, A. and Isaeva, O, V. 2021,P55).*

Proposition 2.1**.** If $\mathcal{U}$ and $\mathcal{V}$ are members of ***M***, then $\mathcal{U}\mathcal{V}$ is an element of ***M***.

Proof: we will prove that $\mathcal{U}\mathcal{V}$ satisfy the definition of varieties.

Let $(\boldsymbol{G}, \varphi) \in \mathcal{U}\mathcal{V}$, then there is an *m*-ideal A of $(\boldsymbol{G}, \varphi)$ such that $A \in \mathcal{U}$ and $\boldsymbol{G}/A \in \mathcal{V}$. For an *m*-subgroup *H* of $(\boldsymbol{G}, \varphi)$, $A \wedge H$ is an *m*-ideal of *H*, and an *m*-subgroup of *A*. Also, $H/(A \wedge H)$ is *m*-isomorphic to AH/A, and AH/A is *m*-subgroup of $\boldsymbol{G}/A$. Consequently, $A \wedge H \in \mathcal{U}$ and $H/(A \wedge H) \in \mathcal{V}$, since $\mathcal{U}$ and $\mathcal{V}$ are varieties. Thus $H \in \mathcal{U}\mathcal{V}$.

Let $\{(\boldsymbol{G}(i), \varphi), i \in I\}$ be element of $\mathcal{U}\mathcal{V}$. then for each $i \in I$, there exist an ideal $H(i)$ of $(\boldsymbol{G}(i), \varphi)$, such that, $H(i) \in \mathcal{U}$ and $\boldsymbol{G}(i)/H(i) \in \mathcal{V}$. Consequently, $\prod H(i) \in \mathcal{U}$ and $\prod \boldsymbol{G}(i)/H(i) \in \mathcal{V}$. Whence, $\prod H(i) \in \mathcal{U}$ and $\prod \boldsymbol{G}(i)/\prod H(i) \in \mathcal{V}$, thus $(\prod \boldsymbol{G}(i), \varphi) \in \mathcal{U}\mathcal{V}$.

Finally, let $(\boldsymbol{G}, \varphi) \in \mathcal{U}\mathcal{V}$, then $(\boldsymbol{G}, \varphi)$ contains an *m*-ideal *A* such that $A \in \mathcal{U}$ and $\boldsymbol{G}/A \in \mathcal{V}$. Let *f* be an *m*-homomorphism of $(\boldsymbol{G}, \varphi)$ onto $(\boldsymbol{G}', \varphi)$, then $\boldsymbol{G}/\ker f$ is *m*-isomorphic to $\boldsymbol{G}'$, the *m*-homomorphic image of *A*, $f(A)$ in $\boldsymbol{G}'$ is an *m*-ideal of $\boldsymbol{G}'$. the quotient group $\boldsymbol{G}'/f(A)$ is an *m*-homomorphic image of $\boldsymbol{G}/A$. Thus, $f(A) \in \mathcal{U}$ and $\boldsymbol{G}'/f(A) \in \mathcal{V}$. Therefore $(\boldsymbol{G}', \varphi) \in \mathcal{U}\mathcal{V}$.

Proposition 2.2. Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be elements of ***M***, then $\mathcal{U}(\mathcal{V}\mathcal{W}) = (\mathcal{U}\mathcal{V})\mathcal{W}$. In the other words, multiplication is associative.

Proof: Let $(\boldsymbol{G}, \varphi) \in \mathcal{U}(\mathcal{V}\mathcal{W})$, then there exists an *m*-ideal *A* of $(\boldsymbol{G}, \varphi)$ such that $A \in \mathcal{U}$ and $\boldsymbol{G}/A \in \mathcal{V}\mathcal{W}$. Since $\boldsymbol{G}/A \in \mathcal{V}\mathcal{W}$, G/A contains an *m*-ideal H/A such that $H/A \in \mathcal{V}$ and $(\boldsymbol{G}/A)/(H/A)$ is m-isomorphic to $G/H \in \mathcal{W}$.

Now, $A \in \mathcal{U}$ and $H/A \in \mathcal{V}$ implies that $H \in \mathcal{UV}$. Since $\boldsymbol{G/H} \in \mathcal{W}$; it must be that $\boldsymbol{G} \in (\mathcal{UV})\mathcal{W}$. Therefore $\mathcal{U}(\mathcal{VW}) \subseteq (\mathcal{UV})\mathcal{W}$.

Conversely, let $\boldsymbol{G} \in (\mathcal{UV})\mathcal{W}$ and let B be the largest convex subgroup of $\boldsymbol{G}$ belonging to $\mathcal{UV}$. Then $\boldsymbol{G}/B \in \mathcal{W}$. Let $\boldsymbol{C}$ be the largest convex subgroup of $\boldsymbol{B}$ belonging to $\mathcal{U}$, then $B/C \in \mathcal{V}$ and C is an m-ideal of $\boldsymbol{G}$. Thus $C \in \mathcal{U}$, $B/C \in \mathcal{V}$ and since $(\boldsymbol{G}/C)/(B/C)$ is m-isomorphic to $\boldsymbol{G}/B \in \mathcal{W}$, we also have $(\boldsymbol{G}/C)/(B/C) \in \mathcal{W}$. Therefore, $\boldsymbol{G} \in \mathcal{U}(\mathcal{VW})$

Proposition 2.3. For elements $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ of $\boldsymbol{M}$, if $\mathcal{U} \leq \mathcal{V}$ then $\mathcal{UW} \leq \mathcal{VW}$ and $\mathcal{WU} \leq \mathcal{WV}$.

Proof: Assume $\mathcal{U} \leq \mathcal{V}$ and $(\boldsymbol{G}, \varphi) \in \mathcal{UW}$. Then, there exist an m-ideal $\boldsymbol{A}$ of $(\boldsymbol{G}, \varphi)$ such that $\boldsymbol{A} \in \mathcal{U}$ and $\boldsymbol{G/A} \in \mathcal{W}$. Since $\mathcal{U} \leq \mathcal{V}$, then $A \in \mathcal{V}$ and thus, $(\boldsymbol{G}, \varphi) \in \mathcal{V}\,\mathcal{W}$. Therefore, $\mathcal{UW} \leq \mathcal{VW}$.

Let $G \in \mathcal{WU}$. Then, there exist an m-ideal $\boldsymbol{A}$ of $\boldsymbol{G}$ such that $\boldsymbol{A} \in \mathcal{W}$ and $\boldsymbol{G/A} \in \mathcal{U}$. Since $\mathcal{U} \leq \mathcal{V}$, then $G/\boldsymbol{A} \in \mathcal{V}$ and thus, $\boldsymbol{G} \in \mathcal{WV}$. Therefore, $\mathcal{WU} \leq \mathcal{WV}$.

By propositions 2.1, 2.2 and 2.3**,** $\boldsymbol{M}$ is a lattice, and a partially ordered semigroup. The semigroup $(\boldsymbol{M}, \cdot)$ has an identity, namely $\boldsymbol{\xi_m}$, since $\boldsymbol{\xi_m}\mathcal{V} = \mathcal{V}$ and $\mathcal{V}\boldsymbol{\xi_m} = \mathcal{V}$ for all $\mathcal{V} \in \boldsymbol{M}$]. (*Giraudet, M. and Rachunek, J. 1999*,P746).

Section 3: Containment Relationship.

Now we will describe some examples of varieties of m-groups.

Definition 3.1: For any ℓ-group $(\boldsymbol{H}, \leq)$, let $H^* = (H, \leq^*)$ denoted by ℓ-group obtained from $\boldsymbol{H}$ by reversing the order; thus $a \leq^* b$ if and only if $b \leq a$**.** Let $\boldsymbol{H}_* = (H_*, \leq)$ denote the ℓ-group obtained from $\boldsymbol{H}$ by reversing group operation $\circ$; that is with group operation $*$ given by $a * b = b \circ a$. Both $\boldsymbol{H}^*$ and $\boldsymbol{H}_*$ are ℓ-groups. *(Huss, M. and Reilly N. 1984)*.

For a variety $\mathcal{V}$ of ℓ-groups, $\mathcal{V}^*$ will denote the variety of those $\boldsymbol{H}^*$ with $\boldsymbol{H}$ in $\mathcal{V}$, in other words, the variety whose defining set of equations is obtained from that of $\mathcal{V}$ either by exchanging $\wedge$ and $\vee$ or by reading the operations from right to left.

Definition 3.2**.** A reversible variety is a variety of ℓ-groups such that $\mathcal{V} = \mathcal{V}^*$.

Huss and Reilly has proved that ℓ-groups varieties $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{R}}$, $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{A}^n}$ for each positive integer n are reversible varieties.

Theorem 3.3**.** Each set of identities defining a reversible variety of ℓ-groups defines a variety of m-groups.

Proof. Let $\mathcal{V} = \mathcal{V}^*$ be a reversible variety of ℓ-groups and let $\boldsymbol{H} \in \mathcal{V}$. Then $\boldsymbol{H}^*$, and hence also $\boldsymbol{H} \times \boldsymbol{H}^*$, satisfy the same ℓ-group identities as $\boldsymbol{H}$. Therefore, the *m*-groups in the form ($\boldsymbol{H}\times\boldsymbol{H}*$, *Exch*), where $\boldsymbol{H}$ is an arbitrary ℓ-group in $\mathcal{V}$, generate a variety of *m*-groups with the same ℓ-identities as $\mathcal{V}$.

Corollary 3.4. The ordered semigroup $\boldsymbol{M}$ of varieties of *m*-groups contains a copy of the set of reversible varieties of all ℓ-groups as a $\wedge$-subsemilattice.

If $\mathcal{V}$ is a reversible variety of ℓ-groups, then the variety of *m*-groups defined by the same ℓ-group identities as $\mathcal{V}$ will be denoted by $\mathcal{V}_m$. A variety $\mathcal{U}$ of *m*-groups will be called an ℓ-variety if $\mathcal{U} = \mathcal{V}_m$ for some variety of ℓ-groups $\mathcal{V}$.

Examples 3.5: $\boldsymbol{\mathcal{A}_m}$, $\boldsymbol{\mathcal{R}_m}$, $\boldsymbol{\mathcal{N}_m}$ and $\boldsymbol{\mathcal{A}^n_m}$ are varieties of *m*-groups.

On any abelian ℓ-group $\boldsymbol{H}$, one can define a mapping ***Inv*** such that for each $a \in \boldsymbol{H}$, Inv(a) = a^{-1}, and that ($\boldsymbol{H}$, ***Inv***) is an *m*-group.

Example 3.6**.** ***The variety I*** is the variety of *m*-groups defined by the identity $\varphi(x) = \boldsymbol{Inv}(x) = x^{-1}$.

Proposition 3.7**.** The variety $\boldsymbol{I}$ is generated by the *m*-group ($\mathbb{Z}$, *Inv*). *(Zenkov, A. (2006), P62)*

Proof. Let ($\boldsymbol{G}$, *Inv*) $\in \boldsymbol{I}$. Clearly, $\boldsymbol{G}$ is abelian, hence it lies in the variety of ℓ-groups $\boldsymbol{\mathcal{A}}$ generated by ($\mathbb{Z}$, $\leq$). The rest follows from the fact that *Inv* is definable in the language of groups.

Theorem 3.8**.** The variety $\boldsymbol{I}$ is the smallest proper variety of *m*-groups ($\boldsymbol{I}$ is cover of $\boldsymbol{\xi_m}$).

Proof. Let $\mathcal{V}$ be a non-trivial variety of ***m***-groups and let $\{e\} \neq (\boldsymbol{G}, \varphi) \in \mathcal{V}$. Take e < $x \in \boldsymbol{G}$ and set y = $x\varphi(x)^{-1}$. Then $\varphi(y) = y^{-1}$, hence the ***m***-subgroup generated by *y* in ($\boldsymbol{G}$, φ) is a copy of ($\mathbb{Z}$, *Inv*), a generating structure for $\boldsymbol{\mathcal{I}}$. Therefore I $\subseteq \boldsymbol{\mathcal{V}}$.

Example 3.9**.** The variety $\boldsymbol{\mathcal{C}}$ is variety of *m*-group defined by the identity $[x, \varphi(x)] = e$.

Example 3.10***:*** Consider the *m*-group ($\boldsymbol{G}_2$, φ) Obviously, $g\varphi(g) = \varphi(g)g$ for each $g \in \boldsymbol{G}_2$, hence ($\boldsymbol{G}_2$, φ) $\in \boldsymbol{\mathcal{C}}$.

Further, set $\boldsymbol{A} = \{a^p{}_{-12}\, a^q{}_{12};\ p, q \in \mathbb{Z}\}$. Then $\boldsymbol{A}$ is a commutative *m*-ideal of ($\boldsymbol{S}_2$, φ) and ($\boldsymbol{S}_2/\boldsymbol{A}$, φ) $\in \boldsymbol{\mathcal{A}_m}$, hence ($\boldsymbol{S}_2$, φ) $\in \boldsymbol{\mathcal{A}^2_m}$.

From this we get the following theorem.

Theorem 3.11 **a)** $\boldsymbol{\mathcal{A}_m}$ is strictly included in $\boldsymbol{\mathcal{C}}$.

b) $\boldsymbol{\mathcal{C} \cap \mathcal{A}_m{}^2 \neq \mathcal{A}_m}$**.**

*proof****: a)*** Follows from the fact that $\boldsymbol{\mathcal{A}_m}$ is a proper subvariety of $\mathcal{C}$.

b) from last example, let ($\boldsymbol{G}$, φ) $\in \mathcal{C}$ and ($\boldsymbol{G}$, φ) $\in \boldsymbol{\mathcal{A}^2_m}$, then ($\boldsymbol{G}$, φ) $\in \mathcal{C} \cap \boldsymbol{\mathcal{A}^2_m}$, and since ($\boldsymbol{G}$, φ) $\notin \boldsymbol{\mathcal{A}_m}$, then $\mathcal{C} \cap \boldsymbol{\mathcal{A}^2_m} \neq \boldsymbol{\mathcal{A}_m}$.

Theorem 3.12. $\boldsymbol{\mathcal{A}_m}$ is the smallest *m*-variety between $\boldsymbol{I}$ and $\boldsymbol{\mathcal{C}}$. (Hence $\boldsymbol{\mathcal{A}_m}$ covers $\boldsymbol{\mathcal{I}}$.). (*Giraudet, M. and Rachunek, J. 1999,*P758).

Proof: Let $\mathcal{V}$ be a variety of m-groups such that $\boldsymbol{I} \subset \mathcal{V} \subseteq \mathcal{C}$. Let $(\boldsymbol{G}, \varphi) \in \mathcal{V}$ be such an m-group that $\varphi \neq Inv$, *i.e.*, that there exists $a \in \boldsymbol{G}$ with $\varphi(a) \neq a^{-1}$. Let us show that there exists an element $e < b \in \boldsymbol{G}$ for which $\varphi(b) \neq b^{-1}$. Let on contrary $\varphi(b) = b^{-1}$ for each $e < b \in \boldsymbol{G}$. Then $a = a^{+} \cdot (a^{-})^{-1}$ implies $\varphi(a^{+}) = \varphi(a) \cdot \varphi(a^{-})$ and thus $(a^{+})^{-1} = \varphi(a) \cdot (a^{-})^{-1}$, that means $\varphi(a) = ((a^{-})^{-1} a^{+})^{-1} = (a^{+}(a^{-})^{-1})^{-1} = a^{-1}$, contradiction.

Hence, consider an element $e < b \in \boldsymbol{G}$ for which $\varphi(b) \neq b^{-1}$. Let $\langle b \rangle \cap \langle \varphi(b) \rangle = \{e\}$ and let $k, p \in \mathbb{Z}$, $\varphi(b)^{k} = b^{p}$, $0 < p$. Then $k < 0$ and $\varphi(b)^{k} = b^{p}$, $b^{k} = \varphi(b)^{p}$, therefore $\varphi(b)^{kp} = b^{p^2} = b^{k^2}$. Consequently $p^2 = k^2$ and so $k = -p$, that means $\varphi(b)^{k} = \varphi(b)^{-p}$. Hence $\varphi(b)^{p} \cdot b^{p} = e$, and since $\boldsymbol{G} \in \mathcal{C}$, $\varphi(b) \cdot b^{p} = e$. This implies $(b) \cdot b = e$, thus $\varphi(b) = b^{-1}$, a contradiction.

Since $(\langle b \rangle, \leq) \cong (\mathbb{Z}, \leq)$ and $(\langle \varphi(b) \rangle, \leq) \cong (\mathbb{Z}^{*}, \leq)$, the subgroup of $\boldsymbol{G}$ generated by $\{b, \varphi(b)$ is an m-subgroup of $(\boldsymbol{G}, \varphi)$ isomorphic to $(\mathbb{Z} \times \mathbb{Z}^{*}, Exch)$. Therefore we have $\mathcal{A}_m \subseteq \mathcal{V}$.

Theorem 3.13**.** $\mathcal{C} \cap \mathcal{R}_m = \mathcal{A}_m$

Proof: Clearly $\mathcal{A}_m \subseteq \mathcal{C} \cap \mathcal{R}_m$. Take $(\boldsymbol{G}, \varphi) \in \mathcal{C} \cap \mathcal{R}_m$. $\boldsymbol{G}$ is a subdirect product of m-groups $(\boldsymbol{G}_i \times \boldsymbol{G}^{*}_i, Exch)$, where each $\boldsymbol{G}_i$ is a totally ordered group, hence for all $g = ((a_i), (b_i)) \in \boldsymbol{G} \cap \prod (\boldsymbol{G}_i \times \boldsymbol{G}_i)$ we have

$$g \cdot \varphi(g) = ((a_i), (b_i)) \cdot (Exch\,(a_i), (b_i)) = ((a_i b_i)\,(b_i a_i)).$$

Since $\boldsymbol{G} \in \mathcal{C}$, $a_i b_i = b_i a_i$.

So all the o-groups $\boldsymbol{G}_i$, and hence the ℓ-group $\boldsymbol{G}$, are in the ℓ-group variety $\mathcal{A}$, therefore the m-group $(\boldsymbol{G}, \varphi)$ belongs to $\mathcal{A}_m$.

Example 3.14. The variety $\mathcal{J}$ is the variety of m-group defined by $\mathcal{J} = \bigcup_{n \in \omega} \boldsymbol{I}^{n}$, the smallest variety of m-groups containing the powers of $\boldsymbol{I}$.

Theorem 3.15**.** $\mathcal{C} \cap \mathcal{J} = \boldsymbol{I}$.

Proof: First we prove that $\mathcal{C} \cap \boldsymbol{I}^{2} = \boldsymbol{I}$. Take $\boldsymbol{G} \in \boldsymbol{I}^{2}$. There is an m-ideal $\boldsymbol{M}$ of $\boldsymbol{G}$ such that $\boldsymbol{M} \in \boldsymbol{I}$ and $\boldsymbol{G}/\boldsymbol{M} \in \boldsymbol{I}$. Since $\boldsymbol{G}/\boldsymbol{M} \in \boldsymbol{I}$, the following identity holds in $\boldsymbol{G}/\boldsymbol{M}$:

$$(g\boldsymbol{M}) \cdot \varphi(g\boldsymbol{M}) = \boldsymbol{M}.$$

In other words, $g \cdot \varphi(g) \in \boldsymbol{M}$ for all $g \in \boldsymbol{G}$, and since $\boldsymbol{M} \in \boldsymbol{I}$,

If, moreover, $\boldsymbol{G} \in \mathcal{C}$, this yields hence $g \cdot \varphi(g) = e$.

Now assume $\mathcal{C} \cap \boldsymbol{I}^{n-1} = \boldsymbol{I}$ for some $n \geq 2$. Then $\boldsymbol{I}\,\mathcal{C} \cap \boldsymbol{I}^{n} = \boldsymbol{I}^{2}$, and hence

$$\mathcal{C} \cap \boldsymbol{I}^{n} = \mathcal{C} \cap \boldsymbol{I}\,\mathcal{C} \cap \boldsymbol{I}^{n} = \mathcal{C} \cap \boldsymbol{I}^{2} = \boldsymbol{I}.$$

These yields

$$\mathcal{C} \cap \mathcal{J} = \mathcal{C} \cap \left(\bigcup_{n \in \omega} \boldsymbol{I}^{n}\right) = \bigcup_{n \in \omega} (\mathcal{C} \cap \boldsymbol{I}^{n}) = \boldsymbol{I}.$$

Corollary 3.16. ***a)*** $\mathcal{A}_m \cap \mathcal{J} = \boldsymbol{I}$.

b) $\mathcal{J}$ is strictly contained in $\mathcal{N}_m$.

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