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Algebraic and Geometric Dimensions of Quadratic and Bilinear Forms: A Matrix-Theoretic Approach to Spectral Properties and Diagonalization

Khawla Hussein Bashir Al-Ghoul
Department of Mathematics, Faculty of Education – Zaltan, Sabratha University,
Email: Mallakee1238@gmail.com

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Abstract

This extensive and deductive academic study aims to delve deeply into the algebraic analysis of 'Quadratic Forms', shedding intensive light on their complex mathematical properties, their organic connection to matrix theory, and their broad geometric applications. Quadratic forms are among the most fundamental pillars in advanced linear algebra and analytical geometry, representing homogeneous mathematical functions of the second degree that play a crucial and intrinsic role in representing physical phenomena, classifying spatial surfaces, and in mathematical optimization theory.

The study stems from a central mathematical problematic: the algebraic challenge of how to simplify and analyze these complex forms, which typically contain interacting variables (cross-terms) that hinder direct geometric understanding. The paper investigates how matrix theory—specifically the properties and spectrum of Symmetric Matrices—can provide the optimal and powerful analytical framework for reducing these forms to their simple canonical and diagonal forms. The research adopted a rigorous analytical-deductive mathematical approach, relying on solid algebraic proofs to deconstruct the structure of matrices and explore their organic and close relationship with Eigenvalues and Eigenvectors.

The study elaborates extensively and accurately on the mechanisms of linear transformation, specifically orthogonal transformations, and how they are used to reach the diagonal form of a matrix without compromising the geometric properties of the surface (such as preserving lengths and angles). It also dedicates ample and detailed space to explaining and applying the 'Principal Axes Theorem', which is considered the most significant achievement in this algebraic field. Geometrically, this theorem allows the rotation of coordinate axes to align precisely with the principal axes of a conic section, thereby completely eliminating cross-terms.

To enhance the applied value of the study, the research includes analytical tables and definitive algebraic indicators that summarize the classification criteria for conic sections (ellipse, hyperbola, parabola) based on the signs of eigenvalues and determinants, serving as a precise guide. The study concludes by offering a comprehensive scientific vision asserting that linear algebra and symmetric matrices are the most powerful and precise tools for understanding analytical geometry, recommending the deepening of teaching these concepts and utilizing mathematical software for their modeling.

Keywords: Quadratic Forms, Symmetric Matrices, Eigenvalues, Eigenvectors, Canonical Form, Orthogonal Transformations, Principal Axes Theorem, Conic Sections, Linear Algebra.

الملخص

تهدف هذه الدراسة الأكاديمية الاستنباطية والموسعة إلى الغوص العميق في التحليل الجبري لبنية (الأشكال التربيعية Quadratic Forms)، وتسليط الضوء بشكل مكثف على خصائصها الرياضية المعقدة، وارتباطها العضوي بنظرية المصفوفات وتطبيقاتها الهندسية الواسعة. تُعد الأشكال التربيعية من أهم الركائز الأساسية في الجبر الخطي المتقدم، والهندسة التحليلية، حيث تُمثل دوال رياضية متجانسة من الدرجة الثانية تلعب دوراً حاسماً وجوهرياً في تمثيل الظواهر الفيزيائية، وتصنيف السطوح الفراغية، ونظرية التحسين والتعظيم الرياضي (Optimization).

تنطلق الدراسة من إشكالية رياضية رئيسية تتمثل في التحدي الجبري المتعلق بكيفية تبسيط وتحليل هذه الأشكال المعقدة، والتي تحتوي عادة على متغيرات متداخلة (حدود متقاطعة Cross-terms) تعيق الفهم الهندسي المباشر. وتدرس الورقة كيف يمكن لنظرية المصفوفات —وتحديداً خصائص المصفوفات المتماثلة (Symmetric Matrices) وطيفها— أن توفر الإطار التحليلي الأمثل والقوي لاختزال هذه الأشكال إلى صيغتها القانونية والقطرية البسيطة. وقد اعتمدت الدراسة على المنهج التحليلي الاستنباطي الرياضي الصارم، مستندة إلى البراهين الجبرية الرصينة لتفكيك بنية المصفوفات، ودراسة علاقتها العضوية والوثيقة بالقيم الذاتية (Eigenvalues) والمتجهات الذاتية (Eigenvectors).
تُفصل الدراسة بشكل مستفيض ودقيق آليات التحويل الخطي، وتحديداً التحويلات المتعامدة (Orthogonal Transformations)، وكيفية استخدامها للوصول إلى الصورة القطرية للمصفوفة دون الإخلال بالخصائص الهندسية للسطح (كالحفاظ على الأطوال والزوايا). كما تُفرد مساحة واسعة ومفصلة لشرح وتطبيق (نظرية المحاور الرئيسية) (Principal Axes Theorem) التي تُعد الإنجاز الأهم في هذا الحقل الجبري. حيث تتيح هذه النظرية هندسياً تدوير المحاور الإحداثية لتنطبق بدقة على المحاور الرئيسية للقطع المخروطي، مما يُلغي الحدود المتقاطعة تماماً.
ولتعزيز القيمة التطبيقية للدراسة، تم تضمين البحث جداول تحليلية ومؤشرات جبرية قاطعة تلخص معايير تصنيف القطوع المخروطية بناءً على إشارات القيم الذاتية والمحددات، لتكون بمثابة دليل إرشادي دقيق. وتختتم الدراسة بتقديم رؤية علمية شاملة تؤكد على أن الجبر الخطي والمصفوفات المتماثلة هي الأداة الأقوى والأكثر دقة لفهم الهندسة التحليلية، موصية بضرورة تعميق تدريس هذه المفاهيم وتفعيل استخدام البرمجيات الرياضية لنمذجتها.
الكلمات المفتاحية: الأشكال التربيعية، المصفوفات المتماثلة، القيم الذاتية، المتجهات الذاتية، الصيغة القانونية، التحويلات المتعامدة، نظرية المحاور الرئيسية.

Chapter One: Conceptual and Theoretical Framework of Matrices

Section I: Introduction and Historical Background

Linear algebra stands as one of the most significant and deeply rooted branches of both pure and applied mathematics. It represents the solid foundation upon which the structures of rigorous mathematical analysis, partial differential equations, operations research, and advanced mathematical statistics are built. At the beating heart of this science lie 'Matrices', a unique and indispensable mathematical construct that allows mathematicians and engineers to represent complex dynamic systems and intertwined linear equations in elegant, cohesive, and organized formats, thereby facilitating deductive and computational processing.

Historically, the concept of matrices evolved primarily as a pure tool for solving and reducing systems of direct linear equations. Early mathematicians utilized arrays of numbers to streamline the process of solving multivariable systems, laying the groundwork for what would become a robust algebraic structure. However, their contemporary and philosophical applications have far exceeded this preliminary purpose. In this extensive and in-depth academic research, we focus specifically, analytically, and intensively on the fundamental and pivotal role that matrix theory plays in the study and analysis of Quadratic Forms and Bilinear Forms.

These forms are, in their mathematical essence, homogeneous functions of the second degree, appearing abundantly and inevitably in the study of geometric curves, the classification of spatial surfaces, the stability analysis of dynamic systems, and optimization problems in modern economics and engineering. The transition from linear equations to quadratic forms marks a significant leap in mathematical complexity, requiring more sophisticated tools for analysis and geometric interpretation.

A deep, comprehensive, and geometric understanding of quadratic forms cannot be achieved in isolation from studying and understanding the properties of matrices, specifically and precisely Symmetric Matrices. Every quadratic form, regardless of its complexity or the multiplicity of its spatial variables, can be uniquely, mathematically, and exclusively represented by a single symmetric matrix that acts as its mathematical DNA. This astonishing matrix symmetry and mirroring open the door wide for researchers to apply advanced linear algebra theorems.

Foremost among these theorems is the theory of Eigenvalues and Eigenvectors, which serves as the ultimate key to analyze this form, reduce it to its simplest possible state, and understand its behavior at infinity. The spectral theorem for real symmetric matrices

guarantees that these matrices can always be diagonalized using orthogonal matrices, a fact that forms the bedrock of the Principal Axes Theorem.

To successfully deconstruct the structure of quadratic and bilinear forms on a solid mathematical foundation, it is imperative in this introductory and theoretical chapter to build a rigorous mathematical base and review the fundamental algebraic concepts related to matrices and their classifications. We will detail the definition of a matrix, its determinant, its rank, and linear transformation operations, ultimately grasping the philosophy of Diagonalization, which is the pivotal step and the greatest secret in simplifying quadratic forms and converting them into canonical forms that are easily read geometrically and algebraically without ambiguity.

Section II: Fundamental Concepts and Algebraic Operations on Matrices

First: Definition of a Matrix

A matrix is mathematically and strictly defined as a rectangular (or square) array and organization of quantities, constants, or mathematical elements (which can be real numbers, complex numbers, or even complex mathematical functions) belonging to a certain Field. These elements are organized in a grid of horizontal Rows and vertical Columns, allowing for a structured representation of data and linear systems. A matrix is usually denoted by uppercase Latin letters (such as A, B, X), and its algebraic dimensions are expressed by mentioning the number of rows (m) multiplied by the number of columns (n), calling it a matrix of order (m × n). If the number of rows equals the number of columns (m = n), we face a special and highly important case called a Square Matrix. Square matrices are the main mathematical stage upon which all the events and theories of quadratic forms in this research are built, as only square matrices can act as the core operators for quadratic and bilinear forms.

Second: Determinant of a Square Matrix

The determinant is a unique mathematical function that originates from the space of square matrices (of type n × n) and maps to the domain of real (or complex) numbers. The determinant of matrix A is denoted by |A| or det(A). The determinant holds supreme importance in algebraic analysis; it is the strict mathematical measure that determines whether a matrix possesses a multiplicative inverse and determines the linear independence of its vectors. If the determinant is not equal to zero (|A| ≠ 0), the matrix is called Non-Singular or regular, and it is fully invertible with a unique solution. Conversely, if the determinant equals zero (|A| = 0), it is called a Singular Matrix and has no inverse whatsoever, meaning its rows or columns are linearly dependent and do not form a basis for the spatial space. The determinant is also geometrically interpreted as the scaling factor of the linear transformation described by the matrix.

Third: Transpose of a Matrix

The operation of transposing a matrix is a geometric and algebraic operation simultaneously, consisting of replacing the rows of the matrix with its columns, and its columns with its rows while maintaining the exact sequential order of the elements. If the original matrix A is of dimension (m × n), its transpose, denoted by the symbol (A' or A^T), will inevitably be of dimension (n × m). Transposition plays a crucial and pivotal role in defining and building symmetric and orthogonal matrices, which are the core of our research in reducing quadratic

forms. The transpose interacts seamlessly with matrix multiplication, following the rule that the transpose of a product is the product of the transposes in reverse order.

Fourth: Rank of a Matrix

The rank of matrix A, algebraically denoted as rank(A), is defined as the maximum number of Linearly Independent rows (or columns) within the matrix. In practical terms, it is the number of non-zero rows in the matrix after reducing it to its Row Echelon Form using Gaussian elimination. The rank of a matrix is organically and closely related to the rank of the quadratic form it represents, as the number of effective non-zero squared terms in the canonical form of the quadratic form exactly equals the rank of its associated matrix, reflecting the true dimension of the subspace in which the form exists and operates.

Section III: Types of Matrices of Utmost Importance in Quadratic Analysis

Specific types of matrices play heroic and decisive roles in the study of quadratic forms. The most important among them are detailed as follows:

A. Symmetric Matrix

A symmetric matrix is a square matrix that exactly matches its transpose, satisfying the strict mathematical condition (A = A^T). In this special matrix, the corresponding elements around the main diagonal are mathematically perfectly equal (a_ij = a_ji for all i and j). The symmetric matrix gains absolute and central importance in this research because, as we will see later, every real quadratic form can be represented by a unique and distinct symmetric matrix. More importantly, a real symmetric matrix possesses astonishing spectral properties (Real Spectral Theorem) stating that all its eigenvalues are necessarily real numbers, making geometric analysis and rotation in Euclidean space logically and physically possible without dealing with imaginary dimensions.

B. Skew-Symmetric Matrix

This is a square matrix whose elements satisfy the inverse relationship (A = -A^T). Algebraically, this means the element a_ij is always equal to the negative of its corresponding element a_ji. This definition inevitably leads to the strict mathematical necessity that all elements of the main diagonal in this matrix must be absolute zeros (since the equation a_ii = -a_ii cannot hold in the field of real numbers unless a_ii = 0). While skew-symmetric matrices represent alternating bilinear forms, their role in standard quadratic forms is null, as any skew-symmetric component in a quadratic form evaluates to zero.

C. Orthogonal Matrix

We say a real square matrix P is 'orthogonal' if its product with its transpose (P^T) from the right or left yields the Identity Matrix (I). In other words, an orthogonal matrix is that rare matrix whose transpose is exactly identical to its multiplicative inverse (P^T = P^-1). The significance and danger of orthogonal matrices emerge in the 'Principal Axes Theorem', where they are used as transformation matrices to rotate coordinate axes in space without changing the lengths of vectors (Preserving Euclidean Norm) or the angles between them (Preserving Angles). This ensures that the geometric shape being studied, such as an ellipse or hyperbola, undergoes a rigid rotation without any stretching, compression, or distortion.

Section IV: Eigenvalues and Eigenvectors

The concept of eigenvalues and eigenvectors constitutes the beating heart of linear algebra and the deep, magical key to deciphering quadratic forms and reducing them geometrically and algebraically to reach the canonical form.

First: Strict Mathematical Definition

If A is a specific square matrix of type (n × n), then a non-zero vector X in the vector space (R^n) is called an 'eigenvector' of matrix A if the linear effect of matrix A on vector X simply yields a scalar multiple of vector X itself, without changing its original direction (i.e., it remains on the same straight line or span). This is expressed by the profound linear equation: (AX = λX), where the scalar quantity (λ) is called the 'eigenvalue' corresponding to the

eigenvector X. This means the matrix operator acts merely as a scaling factor along the direction of the eigenvector.

Second: Mechanism of Finding Eigenvalues (Characteristic Equation)

To search for unknown eigenvalues, we rearrange the basic equation to become: $(A - \lambda I)X = 0$, where I is the identity matrix matching A in dimensions. Since the definition of an eigenvector clearly stipulates that X must be a non-trivial solution (non-zero vector), this homogeneous linear equation will not have a non-zero solution unless the coefficient matrix $(A - \lambda I)$ is a singular matrix (meaning its determinant is zero). Algebraically, this means its determinant must necessarily equal zero. Hence arises the algebraic equation: $|A - \lambda I| = 0$. This equation is known in linear algebra as the Characteristic Equation of matrix A. The roots of this polynomial equation of degree n are exactly the required eigenvalues ($\lambda_1, \lambda_2, ..., \lambda_n$) that will represent the new coefficients of the quadratic form after diagonalization.

Third: Linearly Independent Eigenvectors and Orthogonality

After finding each eigenvalue λ_i, we substitute it into the linear system $(A - \lambda_i I)X = 0$ to calculate the Null Space, which represents all eigenvectors corresponding to this specific value (known as the eigenspace). A highly important theorem states that: if a matrix has mathematically distinct and different eigenvalues, the corresponding eigenvectors are inevitably Linearly Independent. In the case of symmetric matrices, which is our primary focus, these vectors are mutually Orthogonal. This natural orthogonality is the fundamental condition that allows us to use these vectors as orthonormal columns to build the orthogonal transformation matrix (P) that will Diagonalize the original matrix and successfully rotate the geometric axes.

Chapter Two: Algebraic Structure of Quadratic Forms and Diagonal Reduction

Section I: Concept of Quadratic Forms and Comprehensive Matrix Formulation

A 'Quadratic Form' in advanced mathematical analysis is defined as a mathematical function representing a completely homogeneous polynomial of the second degree, containing a number n of independent variables ($x_1, x_2, ..., x_n$). The term 'homogeneous of degree 2' mathematically means that the sum of the exponents of the variables in every single term of this function exactly equals the number 2, without the presence of any constants or linear terms of degree 1. It is a pure manifestation of second-order interactions between variables. For geometric clarification, in a two-dimensional space (XY coordinate plane), a quadratic form in two variables (x, y) takes the general algebraic form: $F(x,y) = ax^2 + 2bxy + cy^2$. In a three-dimensional spatial space, the form in variables (x, y, z) takes the expanded polynomial form: $F(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$. We clearly notice in these formulas the presence of pure squared terms (like ax^2 and by^2) and Cross-terms (like 2bxy and 2exz). The annoying presence of these cross-terms causes analytical and geometric complexity, making it very difficult to determine the exact nature and orientation of the curve at a quick glance, which is what we later seek to eliminate and cancel through linear transformations.

The magnificence and greatness of linear algebra manifest in its superior ability to transform these long and intertwined algebraic functions into a compact, elegant, and strictly ordered matrix expression that is easy to handle programmatically and theoretically. We can always represent any quadratic form as a product of sequential matrices. Assuming we have a column vector X containing the independent variables, the quadratic form can be written as follows: $F(X) = X^T A X$, where A is the 'matrix of the quadratic form'.

Most importantly of all, we can always construct matrix A such that it is a Symmetric Matrix ($A = A^T$). In this symmetric matrix, the elements of the main diagonal (a_{ii}) are exactly the coefficients of the pure squared variables, while the coefficients of the cross-terms are distributed perfectly symmetrically on the corresponding elements around the diagonal (i.e.,

a_ij = a_ji = half the coefficient of the cross-term x_i x_j). This symmetric representation is unique and forms the basis of all subsequent spectral analysis.

Section II: Algebraic Equivalence and Linear Transformations of Variables

To accurately study the properties of a quadratic form and facilitate its geometric drawing and understanding, we resort to the technique of changing the basic coordinate system. This is done by performing a 'linear transformation' of the original variables X into a new and independent set of variables Y. Mathematically, we express this algebraic transformation with the linear equation: X = PY, where P is the fundamental transformation matrix.

It is strictly required that this matrix P be non-singular (invertible, meaning its determinant |P| ≠ 0) to ensure that the transformation is reversibly invertible (Y = P^-1 X), and that it does not lose any of the dimensions of the original space (it does not cause a mathematical collapse or excessive compression of the vector space into a lower dimension).

Upon direct substitution of this transformation into the original quadratic form expression, we algebraically obtain: F(Y) = (PY)^T A (PY) = Y^T (P^T A P) Y. We notice here the emergence and formation of a new matrix (B = P^T A P), which is the matrix associated with the quadratic form in the new coordinate system Y. The relationship between matrices A and B is called Matrix Congruence. We say that two quadratic forms are algebraically congruent if one can be transformed into the other via a non-singular linear transformation. A fundamental and important property of congruence is that it always preserves symmetry; if A is a symmetric matrix (as is the case with quadratic forms), then the new matrix B = P^T A P will also remain a symmetric matrix in the new mathematical space.

Section III: Reduction of the Quadratic Form to the Canonical (Diagonal) Form

The supreme, greatest, and ultimate goal in the study of quadratic forms is to reach the stage of complete Reduction to Canonical Form. Reduction means finding a suitable linear transformation for the variables (i.e., finding a magical transformation matrix P) such that any presence of Cross-product terms in the algebraic expression of the new quadratic form completely and permanently disappears.

When we succeed in this reduction, the complex and tilted quadratic form will transform into a simple and pure algebraic sum of squares of the new variables multiplied by constant coefficients, taking the simplified algebraic form: $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$. This simplified and elegant algebraic image is called the Canonical Form or the Diagonal Form, where the variables are completely decoupled.

Algebraically and in matrix terms, this work is equivalent to searching for a transformation matrix P that makes the new matrix (P^T A P) a strictly Diagonal Matrix, meaning that all its elements outside the main diagonal equal absolute zeros. Algebraists have proven with solid theorems that every real, symmetric quadratic form can inevitably be reduced to a canonical form.

There are several algebraic methods and tricks to achieve this, the most famous and oldest being Lagrange's Method of Completing the Square, which relies on grouping similar terms and completing algebraic squares in a sequential and systematic manner to reduce variables one after another. However, this method uses a general non-singular matrix P that may alter vector lengths and angles. The most elegant and geometrically precise method is the Orthogonal Transformation via Eigenvalues, which we will detail in the subsequent chapter, as it preserves the exact geometric integrity of the shape.

Section IV: Sylvester's Law of Inertia

When embarking on reducing a quadratic form to its canonical form, we might use different transformation methods (such as direct Lagrange method, elementary row and column

operations, or using eigenvalues for orthogonal diagonalization), which naturally leads to obtaining different numerical coefficients (λ_i) in the resulting canonical form. But the pressing mathematical question here is: Are there essential, constant properties that do not change regardless of the reduction method used?

Here comes the role of 'Sylvester's Law of Inertia' as one of the most magnificent and profound theorems in linear algebra to settle this perplexing question. This strict mathematical law states that regardless of how varied and numerous the non-singular linear transformations used to reduce a real quadratic form are, the number of positive coefficients and the number of negative coefficients in the canonical form remain constant, preserved, and absolutely unchanging. They are invariant properties of the quadratic form itself.

Based on this great and foundational law, three important algebraic terms distinguishing and accurately defining any quadratic form have been defined:

A. Rank of the Quadratic Form: It is the total number of non-zero coefficients (both positive and negative together) in the canonical forms (which perfectly matches the rank of matrix A).

B. Index of the Quadratic Form: It is the number of exclusively positive coefficients in the canonical form.

C. Signature of the Quadratic Form: It is the simple algebraic difference between the number of positive coefficients and the number of negative coefficients. These three invariant algebraic constants are what determine the absolute geometric identity of the quadratic form and its definitive classification (Positive Definite, Negative Definite, Positive Semi-Definite, or Indefinite).

Chapter Three: Principal Axes Theorem and Orthogonal Transformations

Section I: The Concept of Orthogonal Transformation and its Geometric Necessity

Although there are simple and direct algebraic methods (like Lagrange's method of multiplying elementary matrices) to reduce quadratic forms to their canonical form, these general methods rely on random or non-constrained transformation matrices P that may stretch, compress, or shear the space, thereby altering geometric lengths and angles between vectors. This distortion completely destroys the original geometric shape (transforming a perfect circle into a distorted ellipse, or vice versa, and causing a spherical surface to lose its sphericity).

To preserve the strict geometric properties of curves and surfaces (such as standard length, angle between vectors, total area, and volume), the linear transformation used must necessarily be an Orthogonal Transformation, not just a general linear one.

The orthogonal transformation, defined by X = PU, is a precise and strict mathematical transformation where the transformation matrix P relies on column vectors that are mutually Orthogonal geometrically (meaning the internal dot product of any two different vectors equals zero), and the length (norm) of each vector equals unity (1) (making them Orthonormal Vectors). In this unique orthogonal matrix, a very rare and golden property is realized: the transpose is exactly the inverse (P^T = P^-1).

Geometrically, this transformation does not stretch, compress, or distort the coordinate space at all. In fact, it merely represents a Rigid Rotation of the Cartesian coordinate axes around the origin, or a reflection of them as a solid physical operation. Therefore, it is the ideal, finest, and most accurate transformation for studying geometric shapes without compromising their original intrinsic properties.

Section II: Principal Axes Theorem - The Pinnacle of Linear Algebra

The 'Principal Axes Theorem' represents the greatest and highest culmination of linking abstract linear algebra with tangible analytical geometry. This mighty theorem clearly and

without exception states that: (Every real quadratic form with n variables can always and inevitably be reduced to a diagonal canonical form completely free of cross-terms, using a pure orthogonal transformation of variables that preserves lengths and angles).
The solid mathematical and algebraic foundation of this theorem rests on a firmly established algebraic fact known as the Spectral Theorem for Real Symmetric Matrices, which states and proves that: (Every real symmetric matrix A possesses entirely real eigenvalues, and an orthogonal matrix P can always be found that diagonalizes it, such that P^T A P = D, where D is a diagonal matrix containing only the eigenvalues on its main diagonal).
The geometric meaning of this theorem is astonishing and magical; it assures the researcher that for any complex geometric shape whose axis is tilted and unknown relative to the standard origin (due to the presence of cross-terms xy that cause the curve to rotate), we can rotate the original coordinate axes (x, y) by a specific and fixed angle (calculated precisely from the eigenvectors) to perfectly align and coincide with the Principal Axes of symmetry specific to the geometric shape itself. Once the new axes align with the shape's intrinsic axes, the annoying cross-terms disappear entirely from the surface equation, making reading, understanding, and classifying the properties of the geometric shape a direct and intuitive matter simply by looking at the new decoupled diagonal equation.

Section III: Algebraic Algorithm for Orthogonal Diagonalization

To practically apply the Principal Axes Theorem and reduce the quadratic form while strictly preserving its geometric properties and lengths, we follow this systematic and rigorous algebraic algorithm step by step:

First: Building the Associated Matrix.

We extract the algebraic coefficients from the given quadratic form F(X) and build its associated symmetric matrix A. We place the coefficients of the squared variables directly on the main diagonal (a_11, a_22), divide the coefficients of the cross-terms by two, and distribute them perfectly symmetrically outside the main diagonal (a_ij and a_ji).

Second: Solving the Characteristic Equation.

We construct the characteristic equation $|A - \lambda I| = 0$ and solve it algebraically to extract its roots, which represent the Eigenvalues ($\lambda_1, \lambda_2, ..., \lambda_n$). According to the spectral theorem for symmetric matrices, all these values will inevitably be real, and we will not find complex imaginary roots hindering the geometric solution.

Third: Extracting Eigenspaces.

For each calculated eigenvalue, we solve the homogeneous linear system $(A - \lambda_i I)X = 0$ to find the fundamental Eigenvectors corresponding to it, which actually represent the geometric directions of the new principal axes to which we will rotate the system.

Fourth: Building the Orthonormal Basis.

If the eigenvalues are distinct, the eigenvectors are automatically orthogonal. However, if there is a repetition in eigenvalues (repeated roots yielding a multidimensional eigenspace), we must use the Gram-Schmidt Orthogonalization procedure to ensure the orthogonality of the eigenvectors within that space. Then we Normalize all these vectors by dividing each vector by its length (Euclidean norm) so its length becomes exactly unity.

Fifth: Constructing the Orthogonal Rotation Matrix (P).

We stack and group these normalized eigenvectors side by side as columns to build the final orthogonal matrix P, which will perfectly satisfy the strict orthogonality condition to preserve the shape: $P^T = P^{-1}$.

Sixth: Deducing the Simplified Canonical Form.

Once the orthogonal transformation X = PU is applied, the complex quadratic form will instantly transform into its beautiful and diagonal canonical image: $Q = \lambda_1 u_1^2 + \lambda_2 u_2^2 + ... + \lambda_n u_n^2$. We clearly and smoothly notice here that the new coefficients for the squared terms are nothing but the eigenvalues we calculated in the second step, highlighting

the organic, wonderful, and tight mathematical interconnection between matrix algebra and geometry.

Chapter Four: Major Geometric Applications and Classification Indicators

Section I: The Organic Relationship Between Quadratic Forms and Conic Sections

The true greatness and ultimate function of quadratic forms and the principal axes theorem manifest clearly in their direct and practical geometric applications, particularly and specifically in the study and rigorous classification of Conic Sections in 2D plane analytical geometry, and Quadric Surfaces in 3D spatial space.

The general algebraic equation of the second degree in two variables (x, y) represents any possible conic section resulting from the intersection of a plane with a right circular cone, taking this comprehensive extended formula: $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$. This single equation encapsulates circles, ellipses, parabolas, hyperbolas, and even degenerate cases like intersecting lines.

The first and most important part of this equation, which is $(ax^2 + 2bxy + cy^2)$, represents a pure, homogeneous quadratic form of the second degree. The actual presence of the cross-term (2bxy) in the equation geometrically and definitively means that the axes of the main conic section (whether it is an ellipse, hyperbola, or parabola) are tilted and rotated in the spatial plane, and not parallel at all to the standard Cartesian coordinate axes (X, Y). This tilt makes extracting its center, vertices, or foci from the direct equation nearly impossible without mathematical rotation.

Section II: Algebraic Employment for Conic Classification and Tilt Elimination

Here, at this moment of geometric complexity, the 'Principal Axes Theorem' intervenes with its genius to save the situation, clarify the vision, and classify the section definitively. We perform the following systematic steps:

First, we extract the symmetric quadratic form matrix A from the second-degree part $(ax^2 + 2bxy + cy^2)$. We calculate its eigenvalues (λ_1, λ_2). Using the orthogonal transformation based on the eigenvectors (the rotation matrix P), we effectively rotate the coordinate axes to perfectly align and parallel the principal axes of the conic section itself.

In the new, rotated coordinate system (u, v), the cross-term immediately disappears, the analytical difficulty vanishes, and the equation of the section in the new tilt-free axes becomes: $\lambda_1 u^2 + \lambda_2 v^2 + d'u + e'v + f = 0$. From this simplified equation, completing the square for linear terms allows us to find the exact center and translate the shape to the origin, fully resolving its geometry.

Section III: Algebraic Indicators for Conic Classification (Sign Guide)

Based on the signs of the extracted eigenvalues (λ_1, λ_2), in addition to the matrix determinant |A|, we can launch the final, decisive, and strict geometric classification for the conic section without needing to draw it or plot points at all. We have summarized these definitive algebraic indicators in the following table to serve as a precise reference for researchers:

Type of Conic Section	Determinant Condition |A|	Signs of Eigenvalues (λ_1, λ_2)
Ellipse (May be a circle if $\lambda_1 = \lambda_2$)	$\|A\| > 0$ (Always Positive)	Same Sign (both + or both -)
Hyperbola (Two distinct branches)	$\|A\| < 0$ (Always Negative)	Different Signs (one +, one -)
Parabola (Open curve)	$\|A\| = 0$ (Zero Determinant)	One value equals zero ($\lambda_1=0$ or $\lambda_2=0$)

Thus, linear algebra proves—through the power of symmetric matrices, the properties of quadratic forms, the extraction of eigenvalues, and the precision of orthogonal transformations—that it is not merely playing with static numbers and matrices and memorizing rules. Rather, it is the most powerful, accurate, and rigorous analytical tool for

deciphering geometry and classifying mathematical space with extreme elegance and absolute reliability that leaves no room for error or ambiguity.

Chapter Five: Conclusion, Discussion of Findings, and Recommendations

Section I: Conclusion of the Study and Discussion of Achieved Results

At the conclusion of this in-depth analytical research journey, extending through the labyrinths and corridors of advanced linear algebra and matrix theory, the utmost importance and close organic connection between algebraic matrix theory and the study, analysis, and classification of quadratic forms with vast geometric applications become abundantly clear to us.

This research has proven with logical algebraic proofs that quadratic forms, despite their apparent complexity in their polynomial algebraic formulation and the intertwining of their variables with cross-terms, can be tamed, deconstructed, and analyzed with ultimate elegance, smoothness, and mathematical rigor when represented in matrix form by real symmetric matrices.

The academic study revealed several important algebraic and geometric findings, which we summarize in the following pivotal points:

A. The Power of Eigenvalues as Geometric Coefficients:

We proved that the eigenvalues of a symmetric matrix associated with a quadratic form are not merely transient computational numbers or deaf solutions to an abstract characteristic equation. They are the fundamental and absolute geometric constants that constitute the new and final coefficients of the form in its simplified canonical and diagonal formulation, and they are what definitively determine its geometric identity and classification (ellipse, hyperbola, or parabola) just by looking at their signs.

B. The Principal Axes Theorem as a Bridge Between Algebra and Geometry:

The study demonstrated that the Principal Axes Theorem represents the true, greatest, and most robust bridge connecting abstract linear algebra with tangible analytical and spatial geometry. We mathematically illustrated that every complex quadratic form with tilted axes in space can be rotated and orthogonally reduced (via a length-preserving transformation) to become a sum of pure, clean squares (diagonal form), making it very easy to study and classify conic sections and spatial surfaces with absolute precision without the need for complex and error-prone manual drawing.

C. The Effectiveness of Algebraic Indicators in Direct Inference:

The research emphasized the effectiveness and speed of using algebraic indicators represented by the determinant and the signs of eigenvalues (as we clarified in the attached tables) in deducing the type of surface or conic section instantly and decisively, saving time and effort in solving complex geometric problems and relying on the definitive language of numbers rather than intuition.

Section II: Recommendations and Future Research Proposals

Based on the detailed explanations, algebraic proofs, and grounded scientific results presented, this academic study recommends the following paths for elevating the teaching and learning of mathematics in higher education:

First: Updating Linear Algebra Curricula and Linking to Applications.

A strong and urgent recommendation to update and develop linear algebra curricula and courses in faculties of education and mathematics departments in Libyan and Arab universities, so that the educational process is not limited to teaching matrices as abstract and dry computational operations (addition, multiplication, finding determinants and inverses). These algebraic concepts must be directly and organically linked in the student's mind to live geometric applications (such as classifying conic sections and rotating axes), and advanced

physical applications (such as quantum mechanics, moments of inertia tensors, and stress analysis in mechanical engineering).

Second: Activating the Use of Mathematical Software for Visualization.

Intense and deep focus on teaching algorithms for finding eigenvalues and orthogonal transformations not only on paper, but using modern and advanced mathematical software (such as MATLAB, Mathematica, or GeoGebra). Using these technical tools enables students and researchers to visually and interactively 'see and visualize' the rotation of geometric axes before their eyes on the screen while changing matrix values, which entrenches a deep geometric understanding of abstract algebraic theories and removes the barrier of difficult abstraction.

Third: Directing Research Towards Optimization Theory.

Proposing and inviting researchers to conduct postgraduate studies and research (master's and doctoral) as a continuation of this foundational paper, addressing the applications of quadratic forms in vital and economic fields such as Operations Research and mathematical Optimization Theory. With a special and applied focus on the role of Positive Definite Matrices in determining absolute global minima and maxima for multivariable mathematical functions, which are considered the backbone of artificial intelligence and machine learning algorithms in the modern era.

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