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A closed form for Estimating the Two Parameter Beta Distribution

Entisar A. Elgmati[1] & Nadia B. Gregni[2]

[1,2]Department of Statistics, Faculty of Science, University of Tripoli- Tripoli, Libya
e.elgmati@uot.edu.ly

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Abstract

Recently the closed form estimators of the two parameter beta distribution have been proposed and investigated by several authors. This is due to its importance over the maximum likelihood estimates sometimes. In this paper we present new closed form to estimate beta parameters with simple formula. The proposed methods are compared with the maximum likelihood and moments estimates through simulation study. The results showed that the proposed method efficiently estimated the parameters and its performance get closer to the maximum likelihood and the moment estimates as the sample size increases.

Introduction

The two parameters beta distribution is a simple and flexible distribution that models the data in a finite interval. It is used in several areas, for instance, risk analysis, physical phenomena, project management and modelling probabilities (i.e. Bayesian inference, stochastic modelling and proportions). Its properties, theory and applications has been studied extensively in the literature (e.g. Gupta and Nadarajah 2004). Recently, estimating its parameters has been investigated by many authors, where closed form estimators were proposed and compared with the traditional methods, the maximum likelihood (ML) and the method of moment (MM). The closed forms are preferable over the ML due to their practical advantage, as the ML could be computationally expensive in some applications (Xiao et al. 2021). Tamae et al. (2020) applied the idea of Ye-Chen (2017) estimators for the gamma distribution to the beta distribution to obtain some interesting closed-form estimators called score-adjusted estimators and score-adjusted moment estimators. Nawa and Nadarajah (2023) proposed two closed form estimators motivated by Tamae et al. (2020) and derived expressions for asymptotic variances and asymptotic covariance. Also Papadatos (2024) derived the closed-form estimators for the beta distribution motivated by Ye-Chen estimators for the gamma distribution using Stein's identity and *U*-statistics. Chen and Xiao (2025) proposed and investigated novel closed-form point estimators of two types. The first one through using an equation involving the sufficient statistics as the second moment equation, while the second type estimators were derived by solving the score equations from the generalized beta distribution.

In this work, at first a new approach to estimate the two parameters of beta distribution will be presented. Second, a comparison between the new approach and maximum likelihood method and the method of moments will be presented through simulation study.

The new approach

The beta distribution, denoted as $B(\alpha,\beta)$, is a two-parameter distribution with probability density function (pdf).

$$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0<x<1, \quad \alpha,\beta>0$$

Where α and β are the shape parameters and $\Gamma(.)$ is the gamma function.

In this section, we propose two new methods to estimate the parameters α and β.

Method 1: since the expected value of $\frac{1}{X}$ is given by

$$E\left(\frac{1}{X}\right)=\int_0^1 \frac{1}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx=1+\frac{\beta}{\alpha-1}$$

Therefore, β will be

$$\beta=\frac{1-E(X)}{E(X)}\alpha$$

Where

$$E(X)=\frac{\alpha}{\alpha+\beta}$$

Thus, the estimates of α and β will be

$$\hat{\alpha}=\frac{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i}\right)-1}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i}\right)-\frac{1}{\bar{X}}} \qquad \text{and} \qquad \hat{\beta}=\frac{1-\bar{X}}{\bar{X}}\hat{\alpha}$$

Method 2: Now if the expected value of $\frac{1}{1-X}$ is used instead

$$E\left(\frac{1}{1-X}\right)=\int_0^1 \frac{1}{1-x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx=1+\frac{\alpha}{\beta-1}$$

Thus α will be

$$\alpha=\frac{E(X)}{1-E(X)}\beta$$

Therefore, the estimates of α and β will be

$$\hat{\beta}=\frac{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{1-X_i}\right)}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{1-X_i}\right)-\frac{\bar{X}}{1-\bar{X}}} \quad \text{and} \quad \hat{\alpha}=\frac{\bar{X}}{1-\bar{X}}\hat{\beta}$$

Simulation study

This simulation study has been conducted to investigate the performance of the proposed methods in estimating the beta parameters. Samples of size n= 50, 100, 200, 500 and 1000 were simulated from the beta distribution 1000 times with different parameters values $(\alpha,\beta)=\{(2,2),(4,3),(3,4)\}$.

Table (1) shows the estimates of the parameters along with its mean square errors for ML, MM and the proposed methods which will be denoted as M1 and M2. From the table one can notice that the proposed methods (M1 and M2) performed efficiently and almost close to ML and MM especially when the sample size increases. Generally, the MSE decreases as the sample size increases for all methods. We have tried different values of α,β, which is not presented here, and the results were similar.

Table 1: The estimates of the parameters of beta distribution along with the MSE in parentheses for different sample sizes and different values of the parameters.

		$\alpha=2$	$\beta=2$	$\alpha=4$	$\beta=3$	$\alpha=3$	$\beta=4$
N=50	MLE	2.1358	2.1463	4.2576	3.1835	3.1775	4.2209
		(0.2012)	(0.2123)	(0.8790)	(0.4761)	(0.4318)	(0.7719)
	MM	2.1215	2.1315	4.2403	3.1707	3.1675	4.2052
		(0.2152)	(0.2230)	(0.9105)	(0.4958)	(0.4526)	(0.7929)
	M1	2.2756	2.2954	4.3572	3.2708	3.2671	4.3438
		(0.3424)	(0.4055)	(1.0837)	(0.6262)	(0.5562)	(1.0296)
	M2	2.2854	2.2892	4.3924	3.2902	3.2768	4.3482
		(0.3569)	(0.3386)	(1.1429)	(0.6108)	(0.6513)	(1.1152)
n=100	MLE	2.0664	2.0523	4.1273	3.0777	3.0722	4.1138
		(0.0826)	(0.0834)	(0.3783)	(0.1990)	(0.2016)	(0.3689)
	MM	2.0593	2.0455	4.1268	3.0800	3.0660	4.1054
		(0.0878)	(0.0880)	(0.3956)	(0.2114)	(0.2145)	(0.3890)
	M1	2.1584	2.1487	4.2333	3.1619	3.1336	4.1978
		(0.1504)	(0.1691)	(0.5217)	(0.3434)	(0.2809)	(0.5299)
	M2	2.1636	2.1451	4.1680	3.1083	3.1216	4.1780
		(0.1720)	(0.1515)	(0.6103)	(0.2667)	(0.2996)	(0.5270)
N=200	MLE	2.0193	2.0201	4.0593	3.0462	3.0433	4.0594
		(0.0366)	(0.0355)	(0.1557)	(0.0885)	(0.0915)	(0.1690)
	MM	2.0141	2.0146	4.0545	3.0428	3.0404	4.0564
		(0.0396)	(0.0390)	(0.1642)	(0.0949)	(0.0972)	(0.1781)
	M1	2.0807	2.0826	4.0951	3.0742	3.0802	4.1112
		(0.0739)	(0.0794)	(0.2423)	(0.1440)	(0.1380)	(0.2646)
	M2	2.0861	2.0843	4.1046	3.0792	3.0582	4.0786
		(0.0855)	(0.0756)	(0.2477)	(0.1347)	(0.1418)	(0.2472)
N=500	MLE	2.0125	2.0070	4.0226	3.0202	3.0141	4.0146
		(0.0145)	(0.0148)	(0.0624)	(0.0342)	(0.0358)	(0.0638)
	MM	2.0119	2.0063	4.0187	3.0171	3.0144	4.0147
		(0.0160)	(0.0165)	(0.0670)	(0.0367)	(0.0383)	(0.0670)
	M1	2.0426	2.0375	4.0403	3.0341	3.0238	4.0279
		(0.0335)	(0.0362)	(0.0954)	(0.0570)	(0.0550)	(0.1031)
	M2	2.0459	2.0394	4.0483	3.0389	3.0260	4.0295
		(0.0378)	(0.0347)	(0.1044)	(0.0551)	(0.0587)	(0.0983)
N=1000	MLE	2.0062	2.0054	4.0108	3.0066	3.0128	4.0199
		(0.0076)	(0.0075)	(0.0292)	(0.0156)	(0.0183)	(0.0328)
	MM	2.0051	2.0040	4.0103	3.0060	3.0129	4.0202
		(0.0081)	(0.0080)	(0.0307)	(0.0166)	(0.0193)	(0.0337)
	M1	2.0269	2.0262	4.0123	3.0077	3.0194	4.0294
		(0.0207)	(0.0223)	(0.0458)	(0.0260)	(0.0295)	(0.0562)
	M2	2.0282	2.0266	4.0256	3.0173	3.0166	4.0248
		(0.0233)	(0.0213)	(0.0513)	(0.0270)	(0.0300)	(0.0509)

Conclusion

Estimating the parameters of beta distribution recently has been investigated by many authors. Several closed form estimators were proposed and compared in the literature. The closed forms are preferable over the ML in practice as the ML could be computationally expensive in some applications. In this work, simpler closed form approach is presented to estimate the two parameter beta distribution. From the simulation study one can observe that the estimate of the parameters by the two methods were close to ML and MM estimates. In addition, the MSE of the estimates tends to decrease to zero as sample size increases.

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