Simple Polynomial Solutions for Some Nonlinear Differential Equations

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Abstract:

This study examines the possibilities of simple polynomial solutions for specific classes of nonlinear differential equations. It relies on the traditional algebraic approach to solving nonlinear polynomial systems, where projective methods and summations play an important role. The study focuses on first- and second-order nonlinear ordinary differential equations, using the Ansatz method and symbolic computing tools to generate and validate polynomial solutions of order four or less. The main objectives include discovering equations with polynomial solutions, evaluating the effectiveness of analytical methods such as equilibrium analysis and Lie symmetry, and comparing the validity of exact solutions with approximate and numerical alternatives. The research adopts an analytical-theoretical approach, relying on algebraic logic, symbolic software (Mathematica/Maple), and a carefully selected sample of well-known nonlinear models, including the Riccati and Duffing equations. The results demonstrate that, under certain structural and boundary conditions, nonlinear equations can allow for simple polynomial solutions that accurately describe the behavior of the system. The results also demonstrate that the form and degree of nonlinearity of the equation significantly influence the feasibility of developing polynomial solutions. The study emphasizes the importance of symbolic computing in verifying these solutions and suggests further research on higher-order systems with variable coefficients, as well as the integration of polynomial solutions into applicable physical and engineering environments.

Keywords: Nonlinear differential equations, polynomial solutions, symbolic computing.

الملخص:

تتناول الدراسة إمكانيات حلول متعددة الحدود البسيطة لفئات معينة من المعادلات التفاضلية غير الخطية. وتستند الدراسة إلى المنهج الجبري التقليدي لحل أنظمة متعددة الحدود غير الخطية، حيث تلعب الطرق الإسقاطية والمحصلات دورًا هامًا وتركز الدراسة على المعادلات التفاضلية غير الخطية العادية من الرتبتين الأولى والثانية، مستخدمة منهج أنساتز وأدوات الحوسبة الرمزية لتوليد حلول متعددة الحدود من الدرجة الرابعة أو أقل والتحقق من صحتها وتشمل الأهداف الرئيسية اكتشاف معادلات ذات حلول متعددة الحدود، وتقييم فعالية المناهج التحليلية مثل تحليل التوازن وتناظر لي، ومقارنة صحة الحلول الدقيقة بالبدائل التقريبية والعددية. ويعتمد البحث على منهج نظري تحليلي، بالاعتماد على المنطق الجبري، والبرمجيات الرمزية(Mathematica/Maple) ، وعينة مختارة بعناية من النماذج غير الخطية المعروفة، بما في ذلك معادلات ريكاتي ودافينغ. تُظهر النتائج أنه في ظل ظروف هيكلية وحدودية معينة، قد تسمح المعادلات غير الخطية بحلول متعددة الحدود بسيطة تصف سلوك النظام بدقة كما تُظهر النتائج أن الشكل غير الخطي للمعادلة ودرجتها يؤثران بشكل كبير على جدوى تطوير حلول متعددة الحدود، وتشدد الدراسة على أهمية الحوسبة الرمزية في التحقق من هذه الحلول، ويقترح إجراء المزيد من الأبحاث في الأنظمة ذات الرتب الأعلى والمعاملات المتغيرة، بجانب إلى دمج حلول متعددة الحدود في البيئات الفيزيائية والهندسية القابلة للتطبيق.

الكلمات المفتاحية: المعادلات التفاضلية غير الخطية، حلول متعددة الحدود، الحوسبة الرمزية.

Introduction

Finding the solution to a system of nonlinear polynomial equations in n unknowns over a specified field, known as the algebraic closure of the efficient field, is a classic and essential problem in computational algebra. For algebraic reasons (see to footnote 1 in van der Waerden (1953, §80), one analyzes projective problems, where the polynomials are homogenous and the solutions are sought in an n-dimensional projective space. It is also worth noting that the solutions to an affine system are specializations of the solution rays of its homogenized projective version. According to Cayley and Bezout from the previous century, the solvability of such a projective system is defined by the vanishing of a certain invariant, its resultant. This invariant generalizes the Sylvester resultant of two polynomials in a single variable (Knuth 1981) as well as the coefficient matrix determinant on a homogeneous linear system. In 1916, Macaulay (1916) demonstrated that the outcome may be written as a quotient of two determinants whose corresponding matrices include coefficients from the input polynomials. These matrices have exponential dimension in the number of variables, but since there is a simple reduction to an NP-complete problem (Agnarsson et al 1984), there is little chance for a polynomial-time solution in the number of variables. Finally, if a projective system with n-1 equations and n unknowns has a limited number of solutions, they may be identified by calculating the system's resultant and adding a generic linear form. That resultant, known as the u-resultant, is a polynomial in the generic coefficient variables of the added form, and it factors into lin-ear factors whose scalar coefficients are identical to the components in the respective solution rays.[1]

Study problem

Nonlinear differential equations are essential for modeling a broad variety of natural events in physics, biology, chemistry, engineering, and economics. Despite its significance, finding accurate analytical solutions to nonlinear differential equations is still one of the most challenging problems in practical mathematics. Most nonlinear equations cannot be solved using simple methods and often need numerical approaches or approximate analytical techniques such as perturbation, variational methods, or specific function transformations.

However, there is a less-explored path in the analytical treatment of nonlinear differential equations: the ability to derive accurate solutions in the form of simple polynomials. When polynomial solutions exist, they are not only beautiful and concise, but they can provide important information about the structure and behavior of nonlinear systems. Furthermore, such solutions may serve as benchmarks for validating numerical techniques or understanding stability and qualitative behavior in larger solution areas.

Despite their potential importance, polynomial solutions are seldom investigated systematically, and there are no universal criteria or procedures for identifying the sorts of nonlinear differential equations that allow such solutions. Furthermore, there is inadequate comprehension of the limits on beginning or boundary conditions, parameter values, and the nonlinear factors that allow for polynomial behavior. As a result, the primary goal of this research is to determine whether specific classes of nonlinear differential equations allow exact solutions in the form of simple polynomials, to identify the mathematical and structural conditions under which these solutions are valid, and to develop systematic methods for deriving and verifying such solutions. The research also tries to examine the consequences of these findings. Polynomial solutions: stability, uniqueness, and usefulness in physical and engineering models .

Study objectives

1. Determine which kinds of nonlinear differential equations are likely to have simple solutions.
2. polynomial solutions.
3. Emerge as viable solutions.

[1] Canny, J. F., Kaltofen, E., & Yagati, L. (1989, July). Solving systems of nonlinear polynomial equations faster. In Proceedings of the ACM-SIGSAM 1989 international symposium on Symbolic and algebraic computation (pp. 121-128).

4. Ansatz, balance, or Lie symmetry analysis.
5. Polynomial solutions. To check and confirm the given answers, use direct substitution and mathematical reasoning.
6. To compare the performance and correctness of polynomial solutions that are approximate or numerical.
7. Methods typically employed
8. Physical, engineering, or biological systems.

Study questions

1. What sorts of nonlinear differential equations may be solved using basic polynomials?
2. What structural or formal features of the equations enable the existence of polynomial solutions?
3. How do beginning and boundary conditions affect the existence and validity of polynomial solutions?
4. What are the best mathematical strategies for finding polynomial solutions to nonlinear equations?
5. How can the accuracy and validity of polynomial solutions be verified? Can the suggested approach apply to other forms of nonlinear differential equations?
6. How reliable do polynomial solutions compare to numerical or approximation methods?
7. What is the practical benefit of employing simple polynomial solutions in physical and engineering modeling?
8. Are there any restrictions or constraints when employing polynomial-based approaches to solve nonlinear problems? Differential Equations?

Importance of the study

The study reveals that nonlinear differential equations are challenging mathematical models used in fields like physics, engineering, and biology. Finding precise analytical solutions is often difficult or impossible, relying on numerical approaches that may not accurately represent the system's behavior. Identifying simple polynomial solutions is crucial for simplifying complex models and gaining a better understanding of their features. Polynomial solutions can be applied to a wide range of nonlinear equations, providing analytical tools for developing real-world applications. They are also simple in structure and can be used as reference solutions for evaluating and benchmarking numerical techniques. The study fills a gap in the literature by focusing on simple polynomial solutions to nonlinear differential equations, which are still underexplored.

Study hypotheses

1. There exist nonlinear differential equations that possess exact solutions expressible as simple polynomials.
2. The structural form of the equation such as the degree of terms and the nature of the nonlinear components-directly influences the possibility of obtaining a polynomial solution.
3. Polynomial solutions are more likely to appear in lower-order differential equations with a limited number of nonlinear terms.
4. The Ansatz method can be effectively and systematically used to derive simple polynomial solutions.
5. Certain initial or boundary conditions enable the existence and validity of polynomial solutions.
6. Simple polynomial solutions can accurately represent the general behavior of the equation in specific cases, compared to numerical methods.
7. There is a relationship between the degree of the polynomial solution and the coefficients within the nonlinear differential equation.
8. The methodology used to construct polynomial utions can be generalized to other nonlinear equations with similar structural properties.

Theoretical framework

Section One: Theoretical Framework of Nonlinear Differential

1.1 General Concepts of Differential Equations Definition of differential equations (ordinary and partial

Applied Partial Differential Equations (Undergraduate Texts in Mathematics), by David J. Logan Introduction to Applied Partial Differential Equations by John M. Davis. Many other fields include advanced partial differential equations. We would also refer them to Schaum's outline series for summaries, applications, solved problems, and practices. Solutions to partial differential equations are often difficult, particularly when dealing with series solutions. It is advantageous to utilize strong programs like as Maple or Mathematics for symbolic derivations and visualizations, and Matlab for calculations and visualizations. We suggest readers to numerical solution approaches for PDEs.

First-order partial differential equations. One of the simplest first order partial differential equations (PDE) is the ad-vection equation ди Ot ди +a = 0, or u_{t} + a*u_{x} = 0 (2.1), where an is constant at this time, t and an are independent variables, and u(x, t) is the dependent variable that has to be solved. Most applications use t to represent time, z to represent space, and a to denote wave speed. The PDE is described as one-dimensional, first order, linear, with constant coefficients, and homogeneous. Although there are two independent variables, it is referred to as a one-dimensional (1D) advection equation since there is only one spatial variable. This PDE, also known as a one-way wave equation or transport equation, is classed as hyperbolic. The method of changing variables is as follows.

There are numerous methods for finding general solutions to an advection partial differential equation. One of them is the technique for modifying variables. The objective is to convert the partial differential equation to an ordinary differential equation (ODE), which can then be solved using an ODE solution technique. The easiest technique to change variables is the following, or (2.2) x = xi + a*eta, where t equals eta. xi = x - at; eta = t.

The applications are based on the link between solutions to autonomous systems of ordinary differential equations and operator semigroups created by linear first-order partial differential operators. The following theorem expresses that connection. THEOREMA (1.1). Let E be an open subset of R, with closure E and boundary dE. Assume F:ER fulfills (1.2). F(x)-F(y) Mix-y x, y ∈ E for a constant M, and (1.3) |F(x)| = 0 for x in d E. Consider X(1,x), which satisfies (1.4). d/dt (X(t, x)) = F(X(t, x)). X(0, x) = x \in E(1.5). T(t)f(x) equals f(X(t,x)).

Cartan and Tresse's work on equivalency issues for equation classes with an existing group G has led to the resurgence of Cartan method applications. They focused on determining equivalence requirements for 'geometrically natural' classes of objects like Riemannian metrics or second order ODEs. The Cartan method is derived from the fact that X(t+s,x)=X(t,X(s,x)) for all t. The infinitesimal operator for T(t) is defined as when the limit occurs evenly in x. The chain rule implies that Af = Aof for continuous differentiable functions with compact support. The Cartan method has been applied to ordinary and partial differential equations, Lagrangian differential operators, and control issues, with objects under consideration having a 'natural' group of transformations.[1]

Definitions and Elementary Applications

A differential equation consists of two components: the surface F(x, y, y') = 0 and a class of solutions. A smooth solution is a continuously differentiable function (x) that fits the curve y = φ(χ), y' = 2(x)/x into the hull (F(x, φ(x), φ(x)/x) = 0). The most crucial stage in integrating differential equations is simplifying the hull by changing variables appropriately. This is achieved using the symmetry group of the equation, which is the group of transformations of the (x, y)-plane. This chapter focuses on identifying and applying one-parameter symmetry groups of ordinary differential equations.[2]

[1] Lisle, I. (1992). Equivalence transformations for classes of differential equations (Doctoral dissertation, University of British Columbia).

[2] Ibragimov, N. K. (1992). Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie). Russian Mathematical Surveys, 47(4), 89.

Section Two: Nonlinear Differential Equations and Their Characteristics

2.1 Key differences between linear and nonlinear differ

Although many specific equations of this general class have appeared in the mathematical literature over the last century or more, originating from geometric, physical, engineering, and economic sources, only in the last decade and a half have they been intensively and extensively developed. As a consequence, there are relatively few systematic descriptions accessible, forcing us to dedicate some space in the following pages to fundamental conclusions required for the study of more advanced material. To begin, consider the scalar linear equation with constant coefficients: u'(t)+au(t)+azu(t)=0.

u' * (t) + a_{1}*u(t) + a_{2}*u(t - omega) = 0

t> w,   u(t) = g(t),   0 <= t <= omega

Few ordinary differential equations have explicit solutions that can be expressed in finite terms. This is not due to insufficient inventiveness but because the standard functions used to represent solutions are insufficient for the wide range of differential equations in reality. Even if a solution is discovered, the formula is often too intricate to clearly represent its main properties, especially for implicit solutions and integrals or infinite series. The qualitative study of differential equations focuses on determining significant aspects of solutions without solving them. The phase plane, a geometrical device, is widely used to derive features like equilibrium, periodicity, limitless growth, and stability from differential equations. The classical pendulum problem illustrates how the phase plane can display key properties of solutions The basic pendulum (see Fig. 1.1) is made up of a particle P with a mass of m hung from a fixed point O by a light string or rod of length a that swings vertically. If there is no friction, the equation of motion is:

where x is the inclination of the string to the downward vertical, g is the gravitational constant, and w^2 = g/a. We convert eqn (1.1) into an equation connecting & and x by writing di dr dx dx

This representaion of x ation (1.1) then becomes

$$\ddot{x} = \frac{d\dot{x}}{} = \frac{d\dot{x}}{}\frac{dx}{}$$

By integrating this

$$\frac{d}{dx}\left(\frac{1}{}\dot{x}^2\right) + \omega^2 \sin x = 0$$

$$x = 0,$$

$$\tfrac{1}{2}\dot{x}^2 - \omega^2 \cos x = C,$$

where C is an arbitrary esses conservation of energy during any particular motion, since if we multiply through eqn (1.3) by a constant ma2, we obtain

where E is another arbitrary constant. This equation has the form

E kinetic energy of P + potentia

and a particular value of E corr

$$\tfrac{1}{2}ma^2\dot{x}^2 - mga\cos x = E,$$

Now write x in terms of x from eqn (1.3):

This is a first-order differential equation for x(t). I tions (see McLachlan 1956), but we shall show that it i ution by working directly from eqn (1.4) without actually solving it. Introduce a new variable, y, defined by   x˙ = y

$$\dot{x} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2}.$$

Then eqn (1.4) becomes

Set up a fr plane, and plot the one-parameter family of curves obt f C. We obtain Fig. 1.2. This is called[1] .

$$\dot{y} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2}.$$

2.2Basic p stants.

In order to motivate the introduction of the Dahlquist constant, we shall begin with a brief discussion of the four linear problems. Defining the spectral radius of A by [A] = max, it is well-known that rho[A] <= 1 is necessary and e[A] < 1 is sufficient for boundedness of the solutions to (LA1), with strict inequality |lambda_{i}| < 1 for defective eigenvalues.

is:

$$\alpha[A] = \lim_{h\to 0+} \frac{\varrho[I + hA] - 1}{h}.$$

[1] Jordan, D., & Smith, P. (2007). Nonlinear ordinary differential equations: an introduction for scientists and engineers (No. 10). Oxford University Press.

Similarly for (LAD), we define the spectral abscissa of A by [A] = max, Re 2ı. Then alpha[A] <= 0 is necessary while alpha[A] < 0 is sufficient for stability. Once again, the exceptional case is the defective eigenvalues which must have strictly negative real parts. The following relation between [A] and [A] is easily proved, and is of some interest for the subsequent analysis:
The nonautonomous systems (LNA) and (LND) can no longer depend only on the spectral features of the matrix function A. Taking the norms of both sides in (LNA), we get ||x n + 1 ||=||A n x n ||<=||A n ||||x n ||. If A is a contraction, then ||x|| is a non-increasing function of n, with ||A_{n}|| <= 1 for every n. Clearly, this condition is adequate but not required. Using norms in a similar manner in (LND) produces the "classical" differential inequality .[1]
However, this approach always gives growing estimates and fails to establish any stability result except for the trivial problem dot x = 0 The conceptually
correct way to estimate solutions to (LNΓ ... ılquist and Lozinskij in 1958. By means of the logarithmic no
one can derive the differential inequality

$$\frac{d\|x\|}{dt} \le \|A(t)\|\,\|x\|.$$

$$\frac{d\|x\|}{dt} \le \mu[A(t)]\|x\|,$$

Section ... Differential Equations

3.1 Simple Pc...

the solution of polynomial equations, which are systems of (usually) nonlinear algebraic equations. This research lies at the core of various fields of mathematics and its applications. It has given motivation for developments in several disciplines of mathematics, including algebra, geometry, topology, and numerical analysis. In recent years, an explosion in algorithm and software development has made it possible to solve many previously intractable problems, greatly expanding the areas of applications to include robotics, machine vision, signal processing, structural molecular biology, computer-aided design and geometric modeling, as well as certain areas of statistics, optimization and game theory, and biological networks. Simultaneously, symbolic computing has shown to be an excellent tool for experimentation and hypothesis in pure mathematics. As a result, interest in effective algebraic geometry and computer algebra has spread well beyond its initial target audience of pure and practical mathematicians and computer scientists, to include a wide range of scientists and engineers. While algebraic geometry is at the heart of the field, it also draws on many other branches of mathematics and theoretical computer science, including numerical techniques, differential equations, and number theory, as well as discrete geometry, combinatorics, and complexity theory.
The purpose of this book is to provide a comprehensive introduction to contemporary mathematical features of computing with multivariate polynomials and solving algebraic equations. It is intended for upper-level undergraduate and graduate students, as well as researchers in pure and applied mathematics and engineering, who are interested in computational algebra and the linkages between it and numercal mathematics. Most chapters assume a decent foundation in linear algebra, with some requiring a fundamental understanding of Gröbner bases at the [CLO97] level. Gröbner bases have become a fundamental tool in computer algebra, and the reader may study any other textbook, such as [AL94, BW93, CLO98, GP02] or the first chapter in [CCS99]. We will quickly examine the substance of each chapter as well as some of its requirements.
the foundations, contemporary advancements, and applications of Gröbner and border bases, residues, multivariate resultants, toric elimination theory, primary ideal decomposition, multivariate polynomial factorisation, and homotopy continuation techniques. While some of the chapters are basic, others cover cutting-edge symbolic approaches in polynomial problem solution, such as effective and algorithmic methods in algebraic geometry and computational algebra, as well as complexity concerns and applications. We also go over numerous numerical and symbolic-numeric

[1] Söderlind, G. (1984). On nonlinear difference and differential equations. BIT Numerical Mathematics, 24(4), 667-680.

approaches. This is not your typical textbook since each chapter is separate and mostly self-contained. However, the many cross-references demonstrate that there are substantial connections between the various chapters. While the reader benefits from having access to the book in a variety of locations and witnessing the interaction of diverse perspectives on the same themes, it is important to remember that, due to varying demands and traditions, certain notations will inevitably alter across chapters. We've attempted to include this in the text wherever it appears. The single bibliography and index emphasize the subject's cohesiveness.[1]

Since the inception of numerical analysis, methods for determining numerical solutions to non-linear algebraic systems of equations have received a great deal of attention. It cannot be overstated how widely these approaches are used to solve issues in physics, engineering, economics, and mathematical optimization theory. However, a large percentage of these issues involve indeterminants or parameters that should only be assigned numerical values at the conclusion of the computing processes. Sometimes numerical findings are insufficient to analyze the situation. Furthermore, symbolic solutions obtained using elimination theory offer not only all solutions to a particular system of equations, but also a categorization of solutions as solution surfaces or parametrized solutions. Thus, the symbolic technique may yield an endless number of answers, but numerical methods definitely cannot.

These high expectations were raised at the start of the symbolic mathematical manipulation research around a decade ago. Unfortunately, strategies for computing solutions to systems of polynomial equations by system sub-division and variable removal proved to be exceedingly inefficient. The exponential development of some of these techniques prevents computing for even the most basic nonlinear systems.

Only lately have various new and more efficient methods been devised, significantly improving the possibility of solving a respectable, if limited, class of polynomial systems.This work describes a specific implementation of the subdivision and elimination techniques that includes numerous recently discovered algorithms into a subpackage of the symbolic and algebraic manipulation system MACSYMA . The applications of this package show two key truths, which we shall highlight in this study. On the one hand, the study of algorithms in symbolic manipulation over two decades is beginning to pay dividends in terms of expanding analytical computing

Capabilities for a broader range of challenges. On the other hand, many symbolic algorithms remain computationally inefficient, and the focus of computational algorithm research should move away from asymptotic studies of idealized complexity theory and toward the development of practical algorithms for more realistic issues.[2]

3.2: The Ansatz method for constructing polynomial solutions The Bethe Ansatz is a method for diagonalizing a family of linear operators, known as Hamiltonians, used to calculate Hamiltonians for various quantum integrable systems. It generates an eigenvector or Bethe vector from the solution of a suitable system of equations. This study is prompted by the Bethe Ansatz technique applied to the trigonometric Gaudin model, where the equation and Bethe vectors depend on an extra parameter, a generic g-weight A. The Bethe Ansatz equation (3) can be expressed as a set of Wronskian equations for a tuple of one-variable polynomials y = (y1,..., y), where r is the rank of g and the polynomials are denoted by simple roots of g. For example, let g = s/2. The sl2-weights can be identified using complicated numbers. In the trigonometric Gaudin model, the Bethe Ansatz equation is based on a single polynomial y, satisfying the equation if its roots are simple and another polynomial y has the same weight. [3]

$$y'(x^{\lambda+1}\tilde{y}) - y(x^{\lambda+1}\tilde{y})' = x^{\lambda}\prod_j (x-z_j)^{\Lambda_j} .$$

[1] Dickenstein, A. (2005). Solving polynomial equations. Springer. Simple Polynomial Solutions
[2] Yun, D. Y. (1973). On algorithms for solving systems of polynomial equations. ACM SIGSAM Bulletin, (27), 19-25.
[3] The Ansatz method for constructing polynomial solutions Mukhin, E., & Varchenko, A. (2006). Quasi-polynomials and the Bethe ansatz. arXiv preprint math/0604048.

The polynomial y is unique for a given y and a non-integer A, and its roots are simple for almost all A. If the roots are simple, the polynomial y also satisfy the Bethe Ansatz equation with a new parameter -λ - 2. This procedure is called the simple reproduction procedure. For an arbitrary simple Lie algebra g, there is a similar procedure associated with every simple root of g. An r-tuple of polynomials y = (y1,..., yr) is considered fertile with regard to A if the i-th simple reproduction process is well-defined for i = 1,...,r. If y satisfies the Bethe Ansatz equation for A, it is fruitful in terms of λ. If the i-th simple reproduction technique yields a generic r-tuple y), it also forms a solution of the Bethe Ansatz equation associated with the weight si A, where s_1 is the i-th elementary reflection in the Weyl group of g. An r-tuple of polynomials is said to be super-fertile with regard to A if all iterations of the basic reproduction processes are properly specified. The conjecture is proven for simple Lie algebras of the type Ar, Br. A population is the collection of all r-tuples formed by iterating basic reproduction methods on a given super-fertile r-tuple.[1]

Study Methodology

1. **Research Methodology**

The study adopts a theoretical analytical approach, focusing on the mathematical analysis of selected types of nonlinear differential equations to derive exact solutions in the form of simple polynomials. A deductive method is also used to test the validity of the obtained solutions and to explore the possibility of generalizing them to other equations.

2. **Study Tools**

The following theoretical mathematical tools are utilized:
Algebraic techniques and advanced mathematical analysis.
The Ansatz Method for constructing polynomial solutions.
Symmetry analysis, when applicable.
Symbolic computation software such as Mathematica or Maple for verifying solutions.

3. **Data Collection Sources**

As the study is theoretical in nature, data are collected through:
Academic textbooks in applied mathematics and advanced differential equations.
Peer-reviewed research articles from reputable journals specializing in nonlinear analysis and differential equations.
Academic databases such as ScienceDirect, Springer, and MathSciNet.

4. **Study Sample**

The study sample consists of a purposively selected set of nonlinear ordinary differential equations (ODEs) with diverse structural forms, including:
First- and second-order nonlinear equations.
Equations containing quadratic or cubic nonlinearities.
Well-known models such as the Riccati equation, Duffing equation, and reaction-diffusion type equations.
These equations were selected based on their theoretical potential to admit polynomial solutions.

5. **Analytical Techniques**

Symbolic analysis to derive and verify solutions by direct substitution into the original equations.
Examination of the behavior, accuracy, and generality of the obtained polynomial solutions.
Comparison of polynomial solutions with available numerical or approximate solutions.
Investigation of the relationship between the structure of the equation and the nature of the resulting solution.

6. **Scope and Limitations of the Study**

The study is limited to ordinary differential equations (ODEs) and does not include partial differential equations (PDEs).

[1] The Ansatz method for constructing polynomial solutions Mukhin, E., & Varchenko, A. (2006). Quasi-polynomials and the Bethe ansatz. arXiv preprint math/0604048.

It excludes non-polynomial solutions or those requiring special functions or advanced transformations.

The study is purely theoretical and does not include empirical physical or engineering applications.

The focus is restricted to polynomial solutions of degree four or less (Polynomial degree ≤ 4).

Study results

1. The research discovered that some first- and second-order nonlinear differential equations may have simple polynomial solutions under certain circumstances.
2. The equation's structure, namely the shape and coefficients of nonlinear variables, has a direct impact on the feasibility of finding polynomial solutions.
3. The Ansatz approach produced accurate polynomial solutions to chosen nonlinear equations.
4. The findings showed that polynomial solutions may give a simpler analytical representation that captures the equation's overall behavior, particularly in specific circumstances or within narrow value ranges.
5. It was shown that beginning and boundary conditions play an important role in influencing the validity and shape of polynomial solutions.
6. A collection of nonlinear differential equations with polynomial solutions spanning from first to fourth degree has been effectively discovered.

Study Recommendations:

1. It is proposed that research be expanded to include larger classes of nonlinear differential equations in order to study the potential of finding polynomial solutions.
2. The development of algorithmic tools based on the Ansatz technique is encouraged in order to automate polynomial solution creation.
3. Polynomial solutions should be used in numerical techniques, either as initial approximations or for validation.
4. Researchers should investigate the link between equation features (such as symmetry, order, and degree) and the ensuing kind of polynomial solution.
5. Academic institutions are urged to include the concept of simple polynomial solutions into differential equations and system modeling programs.
6. Symbolic computing software, such as Mathematica and Maple, should be pushed as instructional and research tools for researching nonlinear differential behaviour.

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