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حلول الموجة المنفردة لمعادلة فوم الغير الخطية في الفيزياء الرياضية

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Solitary wave solutions of the nonlinear faom equating in mathematical physics

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الملخص:

في هذه الدراسة الحالية، تم استخدام طريقتين تحليليتين؛ هما طريقة معادلة ريكاتي الإسقاطية وطريقة توسعة (G'/G, 1/G) لاستخلاص حلول الموجات المنفردة الجديدة إلى جانب عدة أشكال موجية دقيقة لمعادلة FAOM غير الخطية في ميكانيكا الموائع. تشمل الحلول المستخلصة أشكالًا متعددة مثل الموجات الجرسية، والموجات المضادة للجرس، والموجات الحركية، والموجات المضادة للحركية، بالإضافة إلى الأشكال الدورية. كما تم إجراء تحليل مقارن مع النتائج المنشورة سابقًا. علاوة على ذلك، تم إعداد عدد من التمثيلات البيانية للحلول الدقيقة المستخرجة باستخدام برنامج Maple.

الكلمات المفتاحية: معادلة فوم الغير الخطية، طريقة معادلة ريكاتي الإسقاطية المعممة، طريقة– $\left(\frac{G'}{G},\frac{1}{G}\right)$ الموسعة.

Abstract: In the present study, two analytical approaches—the projective Riccati equation method as well as the $\left(\frac{G'}{G},\frac{1}{G}\right)$ —expansion method—are employed to derive novel soliton solutions along with various exact waveforms of the nonlinear FAOM equation in fluid mechanics. The obtained solutions include bell−shaped, anti−bell, kink, anti−kink, and periodic wave profiles. A comparative analysis with previously established results is also undertaken. Furthermore, several graphical representations of the derived exact solutions have been produced with the aid of Maple software.

Keywords: Nonlinear foam equation; generalized projective Riccati equation method; the $\left(\frac{G'}{G},\frac{1}{G}\right)$ − expansion method.

Introduction

Investigating precise solitary wave solutions to nonlinear partial differential equations (PDEs) is crucial for understanding nonlinear physical phenomena. Nonlinear waves are ubiquitous across diverse scientific disciplines, particularly within physics, manifesting in areas such as fluid mechanics, body tissue physics, optical fibers, and atomic science. In recent years, numerous robust analytical methods have been developed to determine solitary and periodic wave solutions of nonlinear partial differential equations (PDEs). These include, but are not limited to, the exp(-ϕ(ξ)) expansion method [1−6], the extended auxiliary equation method [7,8], the new generalized (G'/G) expansion method [9−11], the generalized projective Riccati equation method [12−17], and the generalized Riccati equation method [23]. Notably, Conte and Musette [12] introduced an indirect method for identifying solitary wave solutions of certain nonlinear PDEs, which can be expressed as polynomials in two fundamental functions satisfying a projective Riccati equation [24, 25]. This approach has been successfully applied to a variety of nonlinear PDEs, yielding solitary wave solutions that follow specific forms [13−17]. More recently, Yan [16] has proposed a notable generalization of Conte and Musette's method.

The article aims to determine the generalized projective Riccati equations method [13−17] and the $\left(\frac{G'}{G},\frac{1}{G}\right)-$ expansion method [23] for finding the solitons of the following nonlinear faom equation [5]:

$$-u_t+\frac{1}{2}uu_{xx}+2u^2u_x+(u_x)^2=0, \qquad (1.1)$$

Where $u=u(x,t)$ is a main nonlinear evolution equation in the direction of the study of the seepage of liquid foams. These systems are of ultimate direct and effective algebraic orders for verdict the exact solutions, the alone wave solutions and the concerning manipulation of numbers function answers of nonlinear PDEs in mathematical physics.

Therefore, this study is organized in this manner: In Sec. 2, the description of projective Riccati equations plan’s statement. In Sec. 3, writing $\left(\frac{G'}{G},\frac{1}{G}\right)$−expansion method. In Sec. 4, we judgment the exact solution of Eq. (1.1) by using $\left(\frac{G'}{G},\frac{1}{G}\right)$−expansion method. In Sec. 5, we present figures to few answers of Eq.(1.1). In Sect. 6, acquired conclusion.

2. Description of the generalized projective Riccati equations method

The model is based on the nonlinear PDE:

$$P(u,u_x,u_t,u_{xx},u_{tt},\dots)=0, \qquad (2.1)$$

place $u = u(x,t)$ is a mysterious function, P is a polynomial in $u(x,t)$ and allure incomplete descendants at which point the capital order descendants and nonlinear topmost conditions are complicated. We now outline the principal steps of the generalized projective Riccati equation method. $[13-17]$:

Step 1. The wave transformation adopted in this study is given as follows:

$$u(x,t) = u(\xi), \qquad \xi = kx + \omega t, \tag{2.2}$$

to weaken Eq.(2.1) to the giving nonlinear ODE:

$$H(u, u', u'', \dots) = 0, \tag{2.3}$$

Where ω represents speed of the diffusion, H is a polynomial of u(ξ) are total products $u'(\xi), u''(\xi), \dots$ and $' = \frac{d}{d\xi}$.

Step 2. It is assumed that the solution of Eq. (2.3) takes the following form:

$$u(\xi) = \alpha_0 + \sum_{i=1}^{N} \sigma^{i-1}(\xi)[\alpha_i \sigma(\xi) + \beta_i \tau(\xi)], \tag{2.4}$$

Where α_0, α_i and $\beta_i (i = 1,2,\dots,n)$ represent constants to be determined, while the functions $\sigma(\xi)$ and $\tau(\xi)$ fulfill the following ODEs:

$$\sigma'(\xi) = \varepsilon \sigma(\xi)\tau(\xi), \tag{2.5}$$

$$\tau'(\xi) = R + \varepsilon \tau^2(\xi) - \mu \sigma(\xi), \qquad \varepsilon = \pm 1, \tag{2.6}$$

Where

$$\tau^2(\xi) = \varepsilon \left(R - 2\varepsilon \mu \sigma(\xi) + \frac{\mu^2 + r}{R\sigma^2(\xi)} \right), \tag{2.7}$$

attending $r = \pm 1$and R, μ are nonzero constants.

If $R = \mu = 0$, before Eq.(2.4) has the established solution:

$$u(\xi) = \sum_{i=0}^{N} A_i \tau^i(\xi), \tag{2.8}$$

where $\tau(\xi)$satisfies the nonlinear ODE:

$$\tau'(\xi) = \tau^2(\xi), \tag{2.9}$$

Step 3. The number Nrepresent positively integer in (2.4) must be determined by establishing a suitable balance between the highest-order derivatives and the leading nonlinear terms in Eq.(2.4).

Step 4. **Employ** (2.4) in addition to Eqs.$(2.5)-(2.7)$ into Eq.(2.3). Collecting all agreements of the same capacity of $\sigma^i(\xi)\tau^i(\xi)$ $(j=0,1,\dots;i=0,1)$. By setting each coefficient to zero, one obtains a system of algebraic equations that can be solved via Maple to identify the principles of $\alpha_0,\alpha_i,\beta_i,\omega,\mu,k$ and R.

Step 5. It is famous $[13-17]$ that Eqs.(2.5),(2.6) grants the following solutions:

Case 1. When $\varepsilon=-1,r=-1,R>0$,

$$\sigma_1(\xi)=\frac{R\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1},\tau_1(\xi)=\frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1},\qquad(2.10)$$

Case 2. When $\varepsilon=-1,r=1,R>0$,

$$\sigma_2(\xi)=\frac{R\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1},\tau_2(\xi)=\frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1},\qquad(2.11)$$

Case 3. When $\varepsilon=1,r=-1,R>0$,

$$\sigma_3(\xi)=\frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1},\tau_3(\xi)=\frac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1},\qquad(2.12)$$

$$\sigma_4(\xi)=\frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1},\tau_4(\xi)=\frac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1},\qquad(2.13)$$

Case 4. When $\varepsilon=1,r=1$,

$$\sigma_5(\xi)=\frac{C}{\xi},\tau_5(\xi)=\frac{1}{\varepsilon\xi},\qquad(2.14)$$

Case 5. When $R=\mu=0$,

where $C$is a nonzero constant.

Step 6. Substituting the values of $\sigma_0,\sigma_i,\beta_i,\omega,\mu,k$ and R in addition to the answers $(2.10)-(2.14)$ into (2.4) the exact solutions has been obtained of Eq.(2.1).

3. Description of the $\left(\frac{G'}{G},\frac{1}{G}\right)$ –expansion method.

Before outlining the primary steps of this method, it is essential to consider the following remarks (see$[17-22]$):

Remark 1. Considering the second case, order linear ODE:

$$G''(\xi)+\lambda(\xi)=\mu,\qquad(3.1)$$

and set $\phi=\frac{G'}{G},\boldsymbol{\psi}=\frac{1}{G}$, before we receive

$$\phi'(\xi)=-\phi^2(\xi)+\mu\psi(\xi)-\lambda,\qquad\psi'(\xi)=-\phi(\xi)\psi(\xi),\qquad(3.2)$$

where λ and μ are constants while $'=\frac{d}{d\xi}$.

Remark 2. If $\lambda<0$, then the common solution of Eq.(2.1) has the form:

$$\phi'(\xi) = -\phi^2(\xi) + \mu\psi(\xi) - \lambda, \qquad \psi'(\xi) = -\phi(\xi)\psi(\xi),$$

$$G(\xi) = A_1 \sinh(\xi\sqrt{-\lambda}) + A_2 \cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}, \tag{3.3}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma_1 + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{3.4}$$

where $\sigma_1 = A_1^2 - A_2^2$

Remark 3. If $\lambda>0$, therefore the general solution of Eq.(2.1) has the form:

$$G(\xi) = A_1 \sin(\xi\sqrt{\lambda}) + A_2 \cos(\xi\sqrt{\lambda}) + \frac{\mu}{\lambda}, \tag{3.5}$$

and therefore

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma_2 + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{3.6}$$

where $\sigma_2 = A_1^2 + A_2^2$.

Remark 4. If $\lambda = 0$, before the general solution of Eq.(2.1) has the form:

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{3.7}$$

and therefore

$$\psi^2(\xi) = \frac{-\lambda}{\lambda^2\sigma_1 + \mu^2}(\phi^2(\xi) - 2\mu\psi(\xi) + \lambda), \tag{3.8}$$

In the following, we present the main steps of the (G^'/G,1/G)−expansion method [17−22]:

Step 1. Assuming that the solution of Eq.(3.3) maybe articulated by a polynomial in two together variables ϕ and ψ in this manner:

$$u(\xi) = a_0 + \sum_{i=1}^{N}[a_i\phi(\xi) + b_i\psi(\xi)], \tag{3.9}$$

where $a_0, a_i$and $b_i(i = 1,2, \dots, N)$are constants expected driven later satisfying $a_N^2 + a_N^2. \neq 0$.

Step 2. Determine the positive integer number Nin Eq.(3.9) by utilizing the homogeneous balance middle from two points the highest−order derivatives and the maximal nonlinear agreements in Eq.(1.1).

Step 3. Substitute Eq.(3.9) into Eq.(2.3) in addition to (3.2) and (3.4), into the abandoned− help side of Eq.(2.3) maybe convinced into a polynomial in ϕ and ψ, at which point the grade of ψ does not exceed one. Equating each of its coefficients to zero yields a system of algebraic equations, which can then be solved using symbolic computation software such as Maple to determine the values of $a_i, b_i, \omega, \mu, A_1, A_2$ and λ where $\lambda < 0$.

Step 4. Similar to step 4, substitute Eq.(3.9) into Eq.(2.3) in addition to Eq.(3.2) and Eq.(3.6) for $\lambda > 0$, (or (3.2) and (3.8) for $\lambda = 0$), we get the exact solutions of Eq. (2.3) expressed by trigonometric functions (or by rational functions) respectively.

4. Solitons and added exact wave solutions to the nonlinear foam drainage equation

In this section, we ask the generalized projective Riccati equations method and $\left(\frac{G'}{G}, \frac{1}{G}\right)$−expansion method described in Sec. 2 to find many new solitons and different exact wave solutions of Eq.(1.1).

In order to resolve Eq.(1.1), we use the wave transformation (2.2) for converting Eq.(1.1) to the following nonlinear ODE:

$$-u'\omega + \frac{1}{2}uu''k^2 + 2u^2u'k + u^2k^2 = 0, \qquad (4.1)$$

where k and ω are constants, aforementioned that ω is the velocity of the soliton, ω is the frequency of the soliton, k is the wave, and place $u(\xi) = u(kx + \omega t)$.

4.1. On resolving Eq.(1.1) using the method of section 2.

Balancing uu''accompanying u^2u'in Eq.(4.1), therefore the following relation is attained:

$$N + N + 2 = 2N + N + 1 \implies N = 1, \qquad (4.1.1)$$

Since the balance number N is number, therefore we take the formal resolution:

$$u(\xi) = \alpha_0 + \alpha_1\sigma(\xi) + \beta_1\tau(\xi), \qquad (4.1.2)$$

where α_0, α_1 and $\beta_1$are constants expected determined specific that $\alpha_0 \neq 0$ or $\beta_1 \neq 0$.Substituting (4.1.2) into Eq.(4.1) and using (2.5)−(2.6), the abandoned−help side of Eq.(4.1) enhances a polynomial in $\sigma(\xi)$ and $\tau(\xi)$.Setting the coefficients of these expected zero yields the following method of algebraic equations:

[illegible] : $\frac{2}{R^2}kr^2$ [illegible] $\frac{1}{R}k^2r$ [illegible] $\frac{2}{R^2}k$ [illegible] $\frac{1}{R^2}k^2r^2$ [illegible] $\frac{1}{R}k^2$ [illegible] $\frac{1}{R^2}k^2$ [illegible] $\frac{4}{R^2}kr$ [illegible]

[illegible] $\frac{6}{R}k$ [illegible] $\frac{2}{R^2}k^2r$ [illegible] $\frac{6}{R}kr$ [illegible] 0,

[illegible] : $2k^2$ [illegible] $2k$ [illegible] $\frac{1}{2}k^2$ [illegible] $\frac{8}{R}k$ [illegible] $12k$ [illegible] $\frac{3}{2R}k^2$ [illegible] $\frac{4}{R}k^2$ [illegible]

[illegible] $\frac{2}{R}k$ [illegible] $\frac{3}{2R}k^2r$ [illegible] $\frac{4}{R}k^2r$ [illegible] $\frac{2}{R}kr$ [illegible] $\frac{1}{R}k^2$ [illegible] $\frac{8}{R}kr$ [illegible] $\frac{1}{R}k^2r$ [illegible]

[illegible] $\frac{8}{R}k$ [illegible] $\frac{8}{R}kr$ [illegible] 0,

[illegible] $\frac{6}{R}k$ [illegible] $2k$ [illegible] $\frac{2}{R}k^2$ [illegible] $\frac{6}{R}kr$ [illegible] $\frac{2}{R}k^2r$ [illegible] 0,

[illegible] : k^2 [illegible] $\frac{1}{2}Rk^2$ [illegible] $4kr$ [illegible] $4k^2$ [illegible] $6k^2$ [illegible] $6k$ [illegible] Rk^2 [illegible] k^2r [illegible]

$2k^2 r$ [illegible] $2kr$ [illegible] $2Rk$ [illegible] $12k$ [illegible] $\frac{1}{R}$ [illegible] $\frac{1}{2}k^2$ [illegible] $\frac{1}{R}k^2$ [illegible]

$\frac{1}{R}r$ [illegible] $6Rk$ [illegible] $2k^2$ [illegible] $\frac{1}{R}k^2 r$ [illegible] $4k$ [illegible] $16k$ [illegible] $\frac{2}{R}k$ [illegible]

$\frac{2}{R}kr$ [illegible] $0,$

[illegible] $4k$ [illegible] $4k$ [illegible] $2k^2$ [illegible] $12k$ [illegible] $4k^2$ [illegible] $\frac{4}{R}k$ [illegible]

$\frac{1}{R}k^2$ [illegible] $\frac{4}{R}kr$ [illegible] $\frac{1}{R}k^2 r$ [illegible] $0,$

[illegible] $2k^2$ [illegible] $2k$ [illegible] 2 [illegible] $2k^2$ [illegible] $6Rk$ [illegible] $\frac{1}{2}Rk^2$ [illegible] $8Rk$ [illegible]

Rk^2 [illegible] $4Rk$ [illegible] $4k$ [illegible] $\frac{7}{2}Rk^2$ [illegible] $4Rk^2$ [illegible] $8Rk$ [illegible] $0,$

[illegible] 2 [illegible] k^2 [illegible] $2R$ [illegible] k^2 [illegible] $\frac{3}{2}$ [illegible] k^2 [illegible] $\frac{3}{2}R$ [illegible] k^2 [illegible] 2 [illegible] k^2 [illegible] 8 [illegible] k [illegible]

$6R$ [illegible] k [illegible] 2 [illegible] k [illegible] 4 [illegible] $4R$ [illegible] k [illegible] $0,$

[illegible] R [illegible] k^2 [illegible] R [illegible] k^2 [illegible] 2 [illegible] k^2 [illegible] $4R$ [illegible] k [illegible] $4R$ [illegible] k [illegible] $0,$

[illegible] $: k^2$ [illegible] R [illegible] R [illegible] $R^2 k^2$ [illegible] $R^2 k^2$ [illegible] Rk^2 [illegible] $2R^2 k$ [illegible] $2R^2 k$ [illegible] $2Rk$ [illegible]

$2Rk$ [illegible] $0.$ (4.1.3)

According to Step 5, of Section 2, skilled are four cases expected considered in this manner:

Case 1. $(\varepsilon = -1, r = -1)$,we have:

$$\alpha_0 = 0, \alpha_1 = \pm\frac{1}{4}\sqrt{\frac{k(1-\mu^2)}{\omega}}, k = k, R = \frac{4\omega}{k^3}, \beta_1 = \frac{1}{2}k, \tag{4.1.4}$$

From (2.10), $(4.1.2)$ and $(4.1.4)$, we understand the following exact solutions:

$$u(\xi) = \pm\frac{1}{4}k^2\sqrt{\frac{k(1-\mu^2)}{\omega}}\left(\frac{\frac{4\omega}{k^3}\operatorname{sech}\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)+1}\right) + \frac{1}{2}k\left(\frac{\sqrt{\frac{4\omega}{k^3}}\tanh\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)+1}\right), \tag{4.1.5}$$

where $u(\xi) = u(kx + \omega t)$.

Case 2. $(\varepsilon = -1, r = 1)$, we have:

$$\alpha_0 = 0, \alpha_1 = \pm\frac{1}{4}\sqrt{\frac{\mu^2 k + k}{\omega}}, k = k, R = \frac{4\omega}{k^3}, \beta_1 = \frac{1}{2}k, \tag{4.1.6}$$

From (2.11), $(4.1.2)$ and $(4.1.6)$, we understand the following exact solutions:

$$u(\xi)=\pm\frac{\sqrt{\omega}\left(\sqrt{\mu^2+1}\operatorname{csch}\left(2\sqrt{\frac{\omega}{k^3}}\xi\right)+\coth\left(2\sqrt{\frac{\omega}{k^3}}\xi\right)\right)}{\sqrt{k}\left(\mu\operatorname{csch}\left(\sqrt{\frac{\omega}{k^3}}\xi\right)+1\right)}, \tag{4.1.7}$$

where $u(\xi)=u(kx+\omega t)$.

Case 3. $(\varepsilon=1, r=-1)$, leads to the following results:

Result 1. We have

$$\alpha_0=0, \alpha_1=\pm\frac{1}{4}k^2\sqrt{\frac{k\mu^2-k}{\omega}}, k=k, R=-\frac{4\omega}{h^3}, \beta_1=-\frac{1}{2}k, \tag{4.1.8}$$

From (2.12), $(4.1.2)$ and $(4.1.8)$, we understand the periodic solutions of Eq.(1.1) as follows:

$$u(\xi)=-\frac{\sqrt{\omega}\left(\sqrt{\mu^2-1}\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+I\tan\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)\right)}{\sqrt{k}\left(\mu\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \tag{4.1.9}$$

And

$$u(\xi)=-\frac{\sqrt{\omega}\left(\sqrt{\mu^2-1}\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)-I\cot\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)\right)}{\sqrt{k}\left(\mu\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \tag{4.1.10}$$

where $u(\xi)=u(kx+\omega t)$.

Result 2. We have the same values, and from (2.13), $(4.1.2)$ and $(4.1.8)$, we deduce the periodic solutions of Eq. (1.1) as follows:

$$u(\xi)=\frac{\omega\sqrt{\frac{k(\mu^2-1)}{\omega}}\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)-k^2\sqrt{-\frac{\omega}{k^3}}\tan\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)}{k\left(\mu\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \tag{4.1.11}$$

and

$$u(\xi)=\frac{\omega\sqrt{\frac{k(\mu^2-1)}{\omega}}\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+k^2\sqrt{-\frac{\omega}{k^3}}\cot\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)}{k\left(\mu\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \tag{4.1.12}$$

Remark 1. Note that, if $(R=\mu=0)$ therefore we have the trival solution

Case 4. $(\varepsilon=1, r=1)$, we have:

$$\alpha_0 = 0, \alpha_1 = \pm\frac{1}{4}k^2\sqrt{\frac{k(\mu^2+1)}{\omega}}, k = k, R = -\frac{4\omega}{k^3}, \beta_1 = -\frac{1}{2}k, \qquad (4.1.13)$$

From (2.11), $(4.1.2)$ and $(4.1.13)$, we deduce the following exact solutions:

$$u(\xi) = \pm\frac{1}{4\xi}\sqrt{\frac{k(\mu^2+1)}{\omega}k^2C} - \frac{k}{2\omega}, \qquad (4.1.14)$$

where $u(\xi) = u(kx + \omega t)$.

4.2 On solving Eq.(1.1) using the arrangement of section 3.

To this aim, adjust $u''u$ with u^2u' in Eq.(4.1) we have $N = 1$. Therefore, (3.12) reduces to

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \qquad (4.2.1)$$

where $a_i(i = 0,1)$, b_1 are constants expected determined aforementioned that $a_2 \neq 0$ or $b_2 \neq 0$. Nowe, we have two cases to solutions expected conferred as follows:

According to Step 5, to be discussed as follows:

Case 1. If $\lambda < 0$, substituting $(4.2.1)$ and using (3.2)–(3.4) and Eq.(3.3) into Eq.(4.1), the left–hand side of Eq.(4.1) becomes a polynomial in $\sigma(\xi)$ and $\tau(\xi)$. Equating the coefficients of this polynomial to zero yields the following system of algebraic equations:

[illegible] : $k^2a_1^2$ [illegible] $2ka_1^3$ [illegible] $\frac{k^2b_1^2 [illegible] 6ka_1b_1^2}{[illegible]}$ [illegible] 0,

[illegible] : [illegible] $4ka_0a_1^2$ [illegible] $k^2a_0a_1$ [illegible] $k^2a_1b_1$ [illegible] 4 [illegible] $ka_1^2b_1$ [illegible] $4a_0kb_1^2$ [illegible] 0,

[illegible] : $2k^2a_1b_1$ [illegible] $6ka_1^2b_1$ [illegible] $2k$ [illegible] $\frac{b_1^3}{[illegible]}$ [illegible] 0,

[illegible] : $k^2a_1^2$ [illegible] a_1 [illegible] $2ka_0^2a_1$ [illegible] k^2 [illegible] a_1^2 [illegible] $k^2b_1^2$ [illegible] $k^2b_1^2$ [illegible] $6ka_1b_1^2$ [illegible] $\frac{1}{2}k^2$ [illegible] b_1^2 [illegible] $2k$ [illegible] $a_1b_1^2$ [illegible]

$\frac{1}{2}k^2$ [illegible] a_0b_1 [illegible] $4k$ [illegible] $a_0a_1b_1$ [illegible] $2k$ [illegible] a_1^3 [illegible] 0,

[illegible] : $k^2a_0b_1$ [illegible] $\frac{\frac{1}{2}k^2 [illegible] b_1^2 [illegible] 2k [illegible] a_1b_1^2}{[illegible]}$ [illegible] $\frac{3}{2}k^2$ [illegible] a_1^2 [illegible] $2k$ [illegible] a_1^3 [illegible] $8ka_0a_1b_1$ [illegible] 2 [illegible] $\frac{k^2b_1^2 [illegible] 6ka_1b_1^2}{[illegible]}$ [illegible] 0,

[illegible] : $2k^2a_0a_1$ [illegible] $k^2a_1b_1$ [illegible] 4 [illegible] $ka_1^2b_1$ [illegible] $4a_0kb_1^2$ [illegible] $4k$ [illegible] $a_0a_1^2$ [illegible] k^2 [illegible] a_0a_1 [illegible] $4k$ [illegible] $\frac{b_1^3}{[illegible]}$ [illegible]

0,

[illegible] : [illegible] b_1 [illegible] $2ka_0^2b_1$ [illegible] $2k^2a_1b_1$ [illegible] $4k$ [illegible] $a_0a_1^2$ [illegible] $\frac{3}{2}k^2$ [illegible] a_0a_1 [illegible] $4k$ [illegible] $a_1^2b_1$ [illegible] $\frac{3}{2}k^2$ [illegible] a_1b_1 [illegible] $2k$ [illegible] $\frac{b_1^3}{[illegible]}$ [illegible]

2 [illegible] $k^2a_1b_1$ [illegible] 4 [illegible] $ka_1^2b_1$ [illegible] $4a_0kb_1^2$ [illegible] 0,

[illegible] : $2k^2a_0b_1$ [illegible] $\frac{\frac{1}{2}k^2 [illegible] b_1^2 [illegible] 2k [illegible] a_1b_1^2}{[illegible]}$ [illegible] a_1 [illegible] $2k$ [illegible] $a_0^2a_1$ [illegible] $\frac{1}{2}k^2$ [illegible] a_0b_1 [illegible] 2 [illegible]

$\left(k^2 b_1^2 \quad \frac{1}{2}k^2 \, b_1^2 \quad 2k \, a_1 b_1^2 \quad \frac{1}{2}k^2 \, a_0 b_1 \quad 4k \, a_0 a_1 b_1\right) \quad 4k \, a_0 a_1 b_1 \quad 0,$

$: k^2 a_0^2 \quad \frac{}{}\left(k^2 b_1^2 \quad \frac{1}{2}k^2 \, b_1^2 \quad 2k \, a_1 b_1^2 \quad \frac{1}{2}k^2 \, a_0 b_1 \quad 4k \, a_0 a_1 b_1\right) \quad a_1 \quad 2k \, a_0^2 a_1$

$2 \, \frac{\frac{1}{2}k^2 \, b_1^2 \, 2k \, a_1 b_1^2}{} \quad 0,$

(4.2.2)

We take a_0, a_1, b_1, μ and ω using Maple:

$$a_0 = 0, a_1 = \frac{1}{2}k, b_1 = \pm\frac{1}{2}k\sqrt{\frac{-\lambda^2\sigma_1 - \mu^2}{\lambda}}, \mu = \mu, \omega = -\frac{1}{4}k^3\lambda, \quad (4.2.3)$$

From (3.3), (3.4), $(4.2.1)$ and $(4.2.3)$, we deduce the soliton solution of Eq.(1.1) as follows:

$$u(\xi) = \frac{1}{2}\frac{k\sqrt{-\lambda}\left(A_1\cosh\left(\sqrt{-\lambda}\xi\right) + A_2\sinh\left(\sqrt{-\lambda}\xi\right)\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right)}$$

$$+\frac{1}{2}\frac{k\sqrt{\frac{\lambda^2\sigma_1 + \mu^2}{\lambda}}}{A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right)}, \quad (4.2.4)$$

Where $\sigma_1 = A_1^2 - A_1^2$..

Result 1. In particular, by sitting $(A_1 = 0, A_2 \neq 0, \mu = 0)$ in Eq.$(4.2.4)$, we receive:

$$u(\xi) = \frac{1}{2}k\left[\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right) \pm \frac{I\sqrt{\lambda A_2^2}\,\mathrm{sech}\left(\sqrt{-\lambda}\xi\right)}{A_2}\right], \quad (4.2.5)$$

Result 2. While, if $(A_1 \neq 0, A_2 = 0, \mu = 0)$, we have that that solution:

$$u(\xi) = \frac{1}{2}k\left[\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) \pm \frac{I\sqrt{A_1^2}\csc\left(\sqrt{-\lambda}\xi\right)}{A_1}\right], \quad (4.2.6)$$

Case 2. If $\lambda > 0$, we have substituting $(4.2.1)$ and using (3.2)−(3.4) and Eq.(3.3) into Eq.(4.1), the left−hand side of Eq.(4.1) enhances a polynomial in (ξ) and $\tau(\xi)$. Setting the coefficients concerning this polynomial expected zero yields the following system of algebraic equations:

$: k^2 a_1^2 \quad 2ka_1^3 \quad \frac{k^2 b_1^2 \, 6ka_1 b_1^2}{} \quad 0,$

$: \quad 4ka_0 a_1^2 \quad k^2 a_0 a_1 \quad \frac{}{} \, k^2 a_1 b_1 \quad 4 \, ka_1^2 b_1 \quad 4a_0 k b_1^2 \quad 0,$

$: 2k^2 a_1 b_1 \quad 6ka_1^2 b_1 \quad 2k \, \frac{b_1^3}{} \quad 0,$

$: k^2 a_1^2 \quad a_1 \quad 2ka_0^2 a_1 \quad k^2 \, a_1^2 \quad \frac{}{} \, ^2 b_1^2 \quad ^2 b_1^2 \quad 6ka_1 b_1^2 \quad \frac{1}{2}k^2 \, b_1^2 \quad 2k \, a_1 b_1^2$

$\frac{1}{2}k^2 \, a_0 b_1 \quad 4k \, a_0 a_1 b_1 \quad 2k \, a_1^3 \quad 0,$

[illegible]: $k^2 a_0 b_1$ [illegible] $\frac{\frac{1}{2}k^2 [illegible] b_1^2 [illegible] 2k [illegible] a_1 b_1^2}{[illegible]}$ [illegible] $\frac{3}{2}k^2$ [illegible] a_1^2 [illegible] $2k$ [illegible] a_1^3 [illegible] $8ka_0a_1b_1$ [illegible] 2 [illegible] $\frac{k^2b_1^2 [illegible] 6ka_1b_1^2}{[illegible]}$ [illegible] $0,$

[illegible]: [illegible] $2k^2a_0a_1$ [illegible] $\frac{[illegible]}{[illegible]}$ [illegible] $k^2a_1b_1$ [illegible] 4 [illegible] $ka_1^2b_1$ [illegible] $4a_0kb_1^2$ [illegible] $4k$ [illegible] $a_0a_1^2$ [illegible] k^2 [illegible] a_0a_1 [illegible] $4k$ [illegible] $\frac{b_1^3}{[illegible]}$

[illegible] $0,$

[illegible]: [illegible] b_1 [illegible] $2ka_0^2b_1$ [illegible] $2k^2a_1b_1$ [illegible] $4k$ [illegible] $a_0a_1^2$ [illegible] $\frac{3}{2}k^2$ [illegible] a_0a_1 [illegible] $4k$ [illegible] $a_1^2b_1$ [illegible] $\frac{3}{2}k^2$ [illegible] a_1b_1 [illegible] $2k$ [illegible] $\frac{b_1^3}{[illegible]}$ [illegible]

2 [illegible] $\frac{[illegible]}{[illegible]}$ [illegible] $k^2a_1b_1$ [illegible] 4 [illegible] $ka_1^2b_1$ [illegible] $4a_0kb_1^2$ [illegible] $0,$

[illegible]: $2k^2a_0b_1$ [illegible] $\frac{\frac{1}{2}k^2 [illegible] b_1^2 [illegible] 2k [illegible] a_1b_1^2}{[illegible]}$ [illegible] a_1 [illegible] $2k$ [illegible] $a_0^2a_1$ [illegible] $\frac{1}{2}k^2$ [illegible] a_0b_1 [illegible]

2 [illegible] $\frac{[illegible]}{[illegible]}\left(k^2b_1^2\right.$ [illegible] $\frac{1}{2}k^2$ [illegible] b_1^2 [illegible] $2k$ [illegible] $a_1b_1^2$ [illegible] $\frac{1}{2}k^2$ [illegible] a_0b_1 [illegible] $4k$ [illegible] $\left.a_0a_1b_1\right)$ [illegible] $4k$ [illegible] $a_0a_1b_1$ [illegible] $0,$

[illegible]: $k^2a_0^2$ [illegible] $\frac{[illegible]}{[illegible]}\left(k^2b_1^2\right.$ [illegible] $\frac{1}{2}k^2$ [illegible] b_1^2 [illegible] $2k$ [illegible] $a_1b_1^2$ [illegible] $\frac{1}{2}k^2$ [illegible] a_0b_1 [illegible] $4k$ [illegible] $\left.a_0a_1b_1\right)$ [illegible] a_1 [illegible] $2k$ [illegible] $a_0^2a_1$ [illegible]

2 [illegible] $\frac{\frac{1}{2}k^2 [illegible] b_1^2 [illegible] 2k [illegible] a_1b_1^2}{[illegible]}$ [illegible] $0.$ $\quad (4.2.7)$

We take a_0, a_1, b_1, μ and ω using Maple:

$$a_0 = 0, a_1 = \frac{1}{2}k, b_1 = \pm\frac{1}{2}k\sqrt{\frac{-\lambda^2\sigma_2 - \mu^2}{\lambda}}, \mu = \mu, \omega = -\frac{1}{4}k^3\lambda, \qquad (4.2.8)$$

From (3.5), (3.6), $(4.2.1)$ and $(4.2.8)$, we deduce the soliton solution of Eq.(1.1) as leads to the attends:

$$u(\xi) = \frac{k}{2}\left(\frac{\sqrt{\lambda}\left(A_1\cos(\sqrt{\lambda}\xi) - A_2\sin(\sqrt{\lambda}\xi)\right)}{A_1\sin(\sqrt{\lambda}\xi) + A_2\cos(\sqrt{\lambda}\xi)} + \frac{\sqrt{\frac{\lambda^2\sigma_2 - \mu^2}{\lambda}}}{A_1\sin(\sqrt{\lambda}\xi) + A_2\cos(\sqrt{\lambda}\xi)}\right), \qquad (4.2.9)$$

Where $\sigma_1 = A_1^2 + A_1^2$.

Result 1. Putting $(A_1 = 0, A_2 \neq 0, \mu = 0)$, we receive:

$$u(\xi) = \frac{1}{2}k\sqrt{\lambda}\left[-A_2\tan(\sqrt{\lambda}\xi) \pm \frac{I\sqrt{A_2^2}\sec(\sqrt{\lambda}\xi)}{A_2}\right], \qquad (4.2.10)$$

Result 2. Putting $(A_1 \neq 0, A_2 = 0, \mu = 0)$, In this result we receive:

$$u(\xi) = \frac{1}{2}k\sqrt{\lambda}\left[A_1\cot(\sqrt{\lambda}\xi) \pm \frac{I\sqrt{A_1^2}\csc(\sqrt{\lambda}\xi)}{A_1}\right], \qquad (4.2.11)$$

Case 3. If $\lambda = 0$,, leads to the following results:

$$a_0 = 0, a_1 = a_1, b_1 = b_1, A_1 = A_1, A_2 = -\frac{1}{2}\frac{-A_1^2 a_1^2 + b_1^2}{\mu A_1^2}, \mu = \mu, \omega = 0, k = 2a_1, (4.2.12)$$

Therefore, we have the trivial solution.

5. Graphical likenesses of few solutions

This section presents graphical representations of select exact solutions. The got solution along with periodic solutions both are kink and anti-kink solutions, bell (bright) and anti-bell (dark) are trigonometric solutions and unsociable wave solutions. To exhibit the obtained results, we now present Figures 1–3, which **illustrate** a selection of representative solutions derived in this section. For this purpose, specific parameter values corresponding to particular cases—namely, solutions (4.1.5), (4.1.7), (4.1.9), (4.1.10), (4.1.11), (4.1.12), (4.2.4), and (4.2.9) of the NLS equation with fourth-order dispersion and two-fold power-law nonlinearity (1.1)—have been considered. For clarity and better visualization, the corresponding graphical representations of these solutions are provided below.

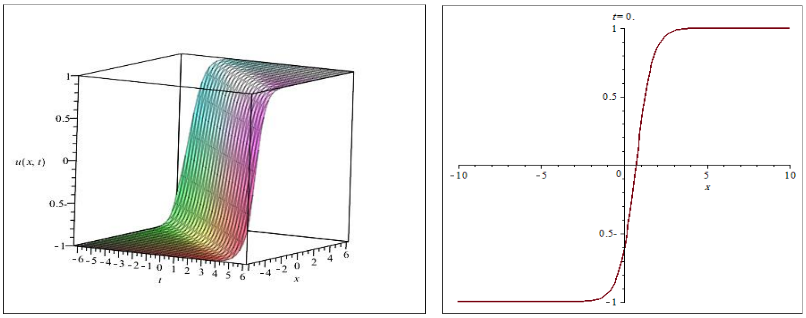


Figure 1: Solution $|u(x,t)|$ of (4.1.5) with $k = 1, \omega = 1, \mu = 2$.

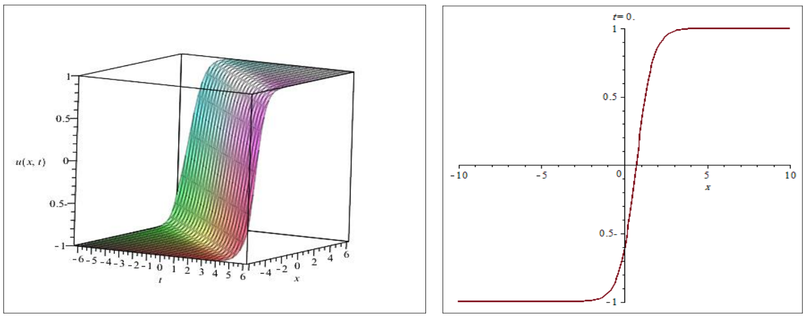


Figure 2: Solution $|u(x,t)|$ of $(4.1.7)$ with $k=1, \omega=1, \mu=2$.

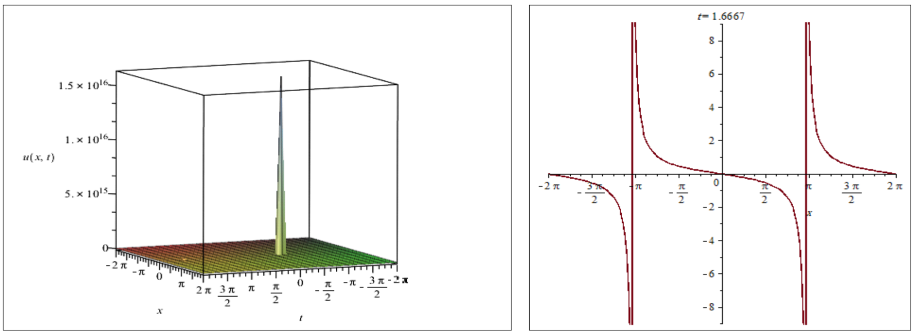


Figure 3: Solution $|u(x,t)|$ of $(4.2.5)$ with $k=1, \omega=1, \lambda=1, \mu=0, A_1=0, A_2=2$.

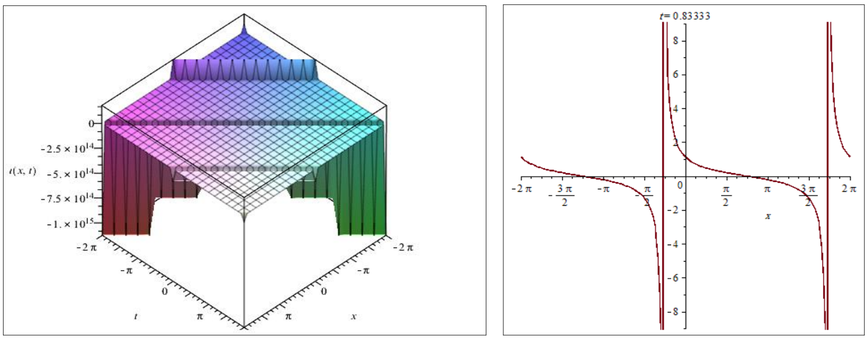


Figure 4: Solution $|u(x,t)|$ of $(4.2.6)$ (the plus signal) with

$k=1, \omega=1, \lambda=1, \mu=0, A_1=2, A_2=0.$

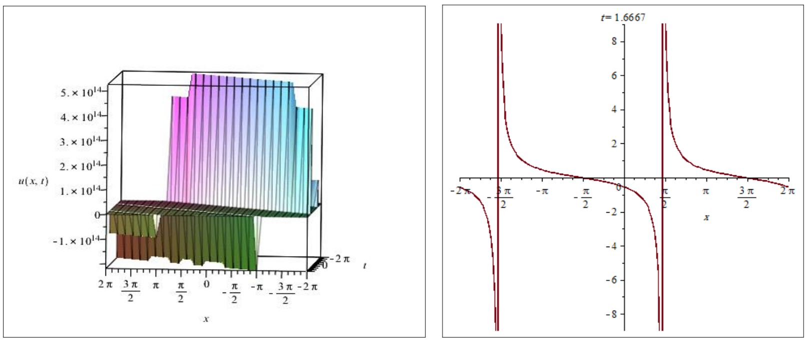
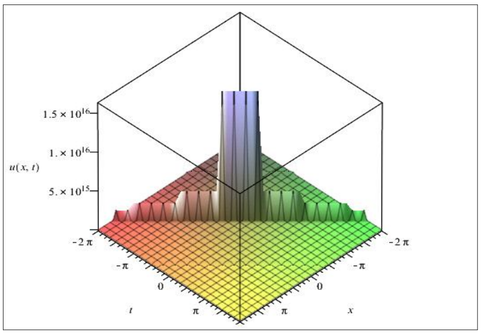


Figure 5: Solution $|u(x,t)|$ of $(4.2.6)$ (the mines signal) with

$$k = 1, \omega = 1, \lambda = 1, \mu = 0, A_1 = 2, A_2 = 0.$$



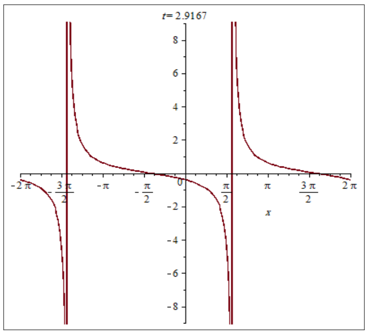


Figure 6:. Plot solution $|u(x,t)|$ of $(4.2.10)$ (the plus signal) with

$$k = 1, \omega = 1, \lambda = 1, \mu = 0, A_1 = 0, A_2 = 2.$$

The graphical representations depict solutions that manifest as both solitary and periodic waves. These figures elucidate the dynamic behavior of the solutions, offering valuable insight into the mechanisms from which such behavior arises

6. Conclusions

This Study applies two different techniques: the projective Riccati equations method and another method is $\left(\frac{G'}{G}, \frac{1}{G}\right)$ –expansion to derive exact wave solutions for the nonlinear foam drainage equation. Our findings are novel and have not been previously published. Moreover, the methods applied in this research have been demonstrated to be powerful and effective tools for determining solitary wave and traveling wave solutions of many other nonlinear equations. In section 5 we present several figures illustrating solution behavior, providing readers with insight into the dynamic characteristics of these solutions. Finally, the accuracy of our results was verified by reintroducing them into the fundamental Equation using Maple.

Conflict of Interests

The authors in relation to this publication have reported no potential conflict of interest.

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