GREEN FUNCTION

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Abstract

The green function theory was developed by the scientist George Green 1793-1841 and is a mathematical function. Where this theory was developed to improve and manage ordinary and partial differential equations with different dimensions and for a time-dependent and time-independent problem .The theory was also developed to serve physics and mechanics, largely in quantum field theory and thermodynamics, as well as in statistical field theory .Thus, during the understanding and analysis of the theory of green function, its role and importance in science were shown, and many applications were presented to understand it, such as i.e. Boundary and Initial Value problem, Wave Equation, Kirchhoff Diffusion Equation, Diffraction Theory, Helmholtz Equation and etc.

So the green function have many important roles in many aspect of sciences, so we try to cover the theory from all of it's aspect.

Introduction:

The green function theory was developed by British scientist George Green, a physicist and mathematician who developed a study on the application of mathematical analysis to electromagnetic theories .The green function theory introduced many basic elements such as the idea of different functions currently used in physics and also clarified the concept of green function Where Greene was the first who was able to create a mathematical theory of electricity and magnetism and his theory formed the basis for the work of many studies for a group of scientists, where the scientist Green works at the same time with the great mathematician Gauss.

Firstly: the problem of study.

The problem of the study lies in identifying the nature and application of green function.

Secondly: Objectives of the study.

1-Identify the concept of green function.

2-Identify the development of green function.

3-Awerness the uses of green function.

Thirdly: importance of green function.

- Theoretical importance:

The green function rule is one of the theoretical rules that researchers find difficult to study it. Therefore, the researcher is trying hard to present the nature of the green function rule and its uses in order for the researchers to obtain a study through which foundations are laid for them in their future studies.

- Practical importance:

The green function rule is one of the rules that are frequently used in applied sciences such as engineering, physics, radiation and mathematics. So The researcher is trying to apply the green function in mathematics.

Fourthly: the methodology of study.

The researcher adopted the experimental method in her study because it is considered one of the most important types of scientific research methods used in applied sciences such as mathematics, physics and engineering..

The researcher also uses the historical method to know the stages of the historical development of the green function theory.

Division of the Study:

-First Topic: definition of green function.

-Second Topic: the idea and concept of green function.

First Topic: Definition of Green Function:

A green function in mathematics is the impulse response to a specified heterogeneous linear differential factor in a domain with certain initial or boundary conditions.

The green function is an integrated kernel that uses all differential equations such as ordinary differential equations in terms of initial or boundary value and also as heterogeneous partial differential with boundary terms. The green function allows visual interpretations of actions associated with a force source or a concentrated charge at a point, making it particularly useful in The field of applied, physical and mechanical mathematics.

This means that if L is the linear differential operator, then

- The Green's function G is the solution of the equation LG = δ, where δ is Dirac's delta function.
- The solution of the initial-value problem Ly = f is the convolution (G δ f), where G is the Green's function.

Through the superposition principle, given a linear ordinary differential equation (ODE), L(solution) = source, one can first solve L(green) = δs, for each s, and realizing that, since the source is a sum of delta functions, the solution is a sum of Green's functions as well, by linearity of L.

The theory was named by this name in relation to the scientist George Green, who was one of the first to develop the concept of the theory in the twenties of the nineteenth century, where the theory was studied through the modern study of linear molecular differential equations.

Under many-body theory, the term is also used in physics, specifically in quantum field theory, aerodynamics, aeroacoustics, electrodynamics, seismology and statistical field theory, to refer to various types of correlation functions, even those that do not fit the mathematical definition. In quantum field theory, Green's functions take the roles of propagators.

Green functions are named after the mathematician and physicist George Green born in Nottingham in 1793 who 'invented' the Green function in 1828. This invention is developed in an essay entitled Mathematical Analysis to the Theories of Electricity and Magnetism originally published in Nottingham in 1828 and reprinted by the George Green Memorial Committee to mark the bicentenary of the birth of George Green in 1993 when he was finally given the recognition he deserved.

The theory is a powerful mathematical tool, not a physical concept, as it was used and applied to classical electromagnetism and acoustics in the late nineteenth century.

Recently, the theory has been a tool for making calculations in particle physics, condensed matter, solid-state physics, quantum mechanics, and other applied mathematics and mathematical physics.

The theory also prompted a revolution in the quantum field in the mid-20th In fact, the Feynman diagram can be considered to be a pictorial representation of a Green function (a Green function associated with wave operators) - what Feynman referred to as a 'propagator.

Second Topic: The idea and concept of green function

One of the fundamental problems of field theory(1) is the construction of solutions to linear differential equations when there is a specified source and the differential equation must be satisfy certain boundary conditions. The purpose of this study is to show how Green's functions provide a strong method for obtaining these solutions. In this study, we present a historical overview of their evolution.

1-Mr. Green's Essay:

In 1828 George Green (1793–1841) published an Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism. In this seminal work of mathematical physics, Green sought to determine the electric potential within a vacuum bounded by conductors with specified potentials In today's notation we would say that he examined the solutions of $\nabla^2 u$-f within a volume V that satisfy certain boundary conditions along the boundary S.

To solve this problem, Green first considered a problem where the source is a point charge. In modern notation, he sought to solve the partial differential equation:

$$\nabla^2 g(r|r_0) = 4\pi\delta(r - r_0), \qquad (1.1.1)$$

where $\delta(r - r_0)$ is the Dirac delta function. We now know that the solution to Equation 1.1.1 is $g = 1/R$, where $R^2=(x- \xi)^2+(y - \eta)^2+(z - \zeta)^2$ Recognizing the singular nature of g, he proceeded as follows:

First Green proved the theorem that bears his name:

$$\iiint_V (\varphi\nabla^2 x - x\nabla^2\varphi)dV = \oiint(\varphi\nabla x - x\nabla\varphi).ndS, \qquad (1.1.2)$$

where the outwardly pointing normal is denoted by n and x and φ are scalar functions that possess bounded derivatives. Then, by introducing a small ball about the singularity at r_0 (because Equation 1.1.2 cannot apply there) and then excluding it from the volume V , he obtained

$$\iiint_V g\nabla^2 u dV + \oiint_S g\nabla u.ndS = \iiint_V u\nabla^2 g dV + \oiint_S u\nabla g.ndS - 4\pi u(r_0) \qquad (1.1.3)$$

because the surface integral over the small ball is $4\pi u(r_0)$ as the radius of the ball tends to zero. Next, Green required that g satisfies the homogeneous boundary condition g=0 along

(1) Any theory in which the basic quantities are fields, such as electromagnetic theory.

the surface S. Since $\nabla^2 u = -f$ and $\nabla^2 g = 0$ within V (recall that the point $r0$ is excluded from V), green found that

$$u(r) = \frac{1}{4\pi} \oiint_S \ \bar{u}\, \nabla \mathrm{g}.\, \mathrm{ndS}, \qquad (1.1.4)$$

when $f = 0$ (Laplace's equation) for any point r within S. Here $\bar{u}$ denotes the value of u on S. This solved the boundary-value problem once g was found Green knew that g had to exist; it physically described the electrical potential from a point charge located at r_0.

Green's essay remained relatively unknown until it was published[2] at the urging of Kelvin between 1850 and 1854. Later Poincar´e[3] summarized our knowledge of Green's functions near the turn of the twentieth century The subsequent evolution of Green's functions can be divided into two parts before and after the publication in 1946 of Methods of Theoretical Physics by P. M. Morse and H. Feshbach.[4] In this study-back version of classnotes that they developed since the late 1930s to teach mathematical methods to physics graduate students, they laid out the four properties that a Green's function must possess. Using the Sturm-Liouville problem given by

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + p(x)y = -q(x), \qquad (1.15)$$

these four properties are:

- The Green's function satisfies the homogeneous differential equation when x not equal $x \neq \xi$, the source point.
- The Green's function satisfies homogeneous boundary conditions.-
- The Green's function is symmetric in the variables x,ξ. -
-The Green's function $g(x|\xi)$ satisfies the condition.

$$\left.\frac{dg}{dx}\right|_{x=\xi+} - \left.\frac{dg}{dx}\right|_{x=\xi} = -\frac{1}{f(\xi)}. \qquad (1.16)$$

[2] Green, G., 1850, 1852, 1854: An essay on the application of mathematical analysis to the theories of electricity and magnetism. J. Reine Angew. Math., 39, 73–89; 44, 356–374;221,161,47.

[3] Poincar´e, H., 1894: Sur les ´equations de la physique math´ematique. Rend. Circ. Mat. Palermo, 8, 57–156.

[4] Morse, P. M., and H. Feshbach, 1946: Methods of Theoretical Physics. MIT Technology Press, 497 pp.

Prior to the publication of Morse and Feshbach's notes, researsher used various tricks to find Green's functions that satisfied these four properties. Morse and Feshbach's great contribution was to show that the "Green's function is the point source solution [to a boundary-value problem] satisfying appropriate boundary conditions." Thus the Green's function could be found by simply solving (in the case of Sturm-Liouville problem).

$$\frac{d}{dx}\left[f(x)\frac{dg}{dx}\right]+p(x)g=-\delta(x-\xi) \qquad (1.1.7)$$

with homogeneous boundary conditions, where $\delta(x-\xi)$ was the recently introduced delta function by Dirac. The advantage of this formulation was that the powerful techniques of eigenvalue expansions and transform methods could be used in a straightforward manner to find Green's functions.

By the 1960's many studies began to champion the use of Green's functions. For example, in Mackie's 1965 book[5] he sought "to give a general account of how certain mathematical techniques, notably those of Green's functions and of integral transforms, can be used to solve important and commonly occurring boundary value problems in ordinary and partial differential equations." In the following sections we turn to the development of Green's functions as they evolved within each general class of differential equations.

2- Potential Equation:

Shortly after the publication of Green's monograph on the European continent, the German mathematician and pedagogue Carl Gottfried Neumann developed the concept of Green's function as it applies to the two-dimensional (in contrast to three-dimensional) potential equation.[6] He defined the two-dimensional Green's function, showed that it possesses the property of reciprocity, and found that it behaves as ln(r) as $r\to\infty$. Using elliptic coordinates he rederived Poisson's integral formula and developed an eigenfunction expansion for the two-dimensional Green's function. In 1875 Paul Meutzner (1849–1914) extended Neumann's work.[7] In particular, he obtained the Green's function for the region within an ellipse (Ellipsenflache) and a circle (Ringfl¨ache). Finally, in his study on the logarithmic

(5) Mackie, A. G., 1965: Boundary Value Problems. Oliver & Boyd, 252 pp.

(6) Neumann, C., 1861: Ueber die Integration der partiellen Differentialgleichung

(7) Meutzner, P., 1875: Untersuchungen im Gebiete des logarithmischen Potentiales. Math. Ann., 8, 319–338. For an alternative derivation, see Sections 15 and 17 in Neumann, C., 1906: Uber das logarithmische Potential. ¨ Ber. Verh. K. Sachs. Ges. Wiss. Leipzig, Math.-Phys. Klasse, 58, 482–559.

potential A. Harnack [8](1851–1888) gave the Green's function for a circle and rectangle.

All of these researcher used a technique that would become one of the fundamental techniques in constructing a Green's function, namely eigenfunction expansions. The investigator would first find an eigenfunction expansion that satisfied both the homogeneous differential equation and boundary conditions The geometry of the problem would determine the coordinate system that was used. Then the Fourier coefficients would be chosen so that the Green's function exhibited the proper behavior (such as 1/r) near the source point. Later on,[9] John Dougall (1867–1960) derived three-dimensional Green's functions in cylindrical and spherical coordinates.

In 1879 Alfred George Greenhill[10] (1847–1927) applied the method of images to construct the Green's function for a rectangular parallelepiped Because his results are expressed as an infinite summation of theta functions it was not very useful and has essentially been forgotten Hector Munro Macdonald[11] (1865–1935) took a slightly different approach in the 1890s. As before, he began with the eigenfunction expansion that satisfied the boundary conditions. But now, the Fourier coefficients were chosen so that the expansion satisfied the general Poisson equation. Then he considered the special case of a point source.

In the late 1890's Arnold Sommerfeld[12] (1868–1951) developed a technique using integration on the complex plane to extend the method of images to several other useful geometries in three dimensions. Ernst William Hobson (1856–1933) then used this method[13] to find the Green's function for a circular disk. Later, Ludwig Waldmann (1913–1980), a young assistant to Sommerfeld, applied this technique in electrostatic calculations of an electron

[8] See Chapter 2 in Harnack, A., 1887: Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene. Leipzig, B. G. Teubner, 170 pp.

[9] Dougall, J., 1900: The determination of Green's function by means of cylindrical or spherical harmonics. Proc. Edinburgh Math. Soc., Ser. 1, 18, 33–83

[10] Greenhill, A. G., 1879: On Green's function for a rectangular parallelepiped. Proc. Cambridge Philos. Soc., 3, 289–293.

[11] Macdonald, H. M., 1895: The electrical distribution on a conductor bounded by two spherical surfaces cutting at any angle. Proc. London Math. Soc., Ser. 1, 26, 156–172Macdonald, H. M., 1900: Demonstration of Green's formula for electric density near the vertex of a right cone. Trans. Cambridge Philos. Soc., 18, 292–297.

[12] Sommerfeld, A., 1897: Uber verzweigte Potentiale im Raum. ¨ Proc. London Math. Soc., Ser. 1, 28, 395–429.

[13] Hobson, E. W., 1900: On Green's function for a circular disc, with applications to electrostatic problems. Trans. Cambridge Philos. Soc., 18, 277–291.

lens.[14]

At the beginning of twentieth century the method of bilinear expansions was developed:

$$g(x, y, z, |\xi, \eta, \zeta) \sum_{n=1}^{\infty} \frac{\psi_n(x, y, z)\psi_n(\xi, \eta, \zeta)}{\lambda_n}, \qquad (1.2.1)$$

Where λ_n and $\psi_n(x, y, z)$ are the nth eigenvalue and eigenfunction, respectively. Adolf Kneser[15] (1862–1930) showed that the Green's function was the symmetric kernel of the integral equation

$$\psi_n(\xi, \eta, \zeta) = \lambda_n \iiint g(x, y, z|\xi, \eta, \zeta)\psi_n\,(x, y, z)dxdydz. \qquad (1.2.2)$$

Assuming that the Green's function can be expressed as an eigenfunction expansion, Equation 1.2.1 follows. As examples, Kneser found the bilinear expansion for rectangular and circular areas and for the surface of a sphere In summary then, by 1950 there were essentially three methods[16] for finding Green functions. The first method simply used a Green's function developed for Helmholtz's equation $\nabla^2 u + k_0^2 u = 0$ and took the limit as $k_0 \rightarrow 0$

The second method wrote the Green's function as a sum of eigenfunctions that satisfied the boundary conditions. The coefficients were then chosen so that the correct singular behavior occurred at the source point. Finally, the third method wrote the Green's function as the sum of the free-space solution plus a harmonic function.[17] The harmonic solution was chosen so that the Green's function satisfied the boundary conditions.

Later on, Kelvin's classic inversion[18] that maps the interior of a circle or sphere to the exterior and vice versa was developed to find the Green's function for Poisson's equation.

Finally Green's functions have been used to solve mixed boundary-value problems involving the two-dimensional Poisson's equation. These problems occur when the boundary

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[14] Waldmann, L., 1937: Zwei Anwendungen der Sommerfeld'schen Methode der verzweigten Potentiale. Phys. Z., 38, 654–663.

[15] Kneser, A., 1911: Integralgleichungen und ihre Anwendungen in der mathematischen Physik. Braunschweig, 293 pp.

[16] See, for example, Bouwkamp, C. J., and N. G. de Bruijn, 1947: The electrostatic field of a point charge inside a cylinder, in connection with wave guide theory. J. Appl. Phys,. 577,562,18 .This paper is of particular note because of its use of the modern definition of the delta function. See their Equation 41.

[17] Weber, E., 1939: The electrostatic potential produced by a point charge on the axis of a cylinder. J. Appl. Phys., 10, 663–666.

[18] Thomson, W., 1845: Extrait d'une lettre de M. William Thomson ` a M. Liouville. J. Math. Pures Appl., 10, 364–367; Thomson, W., 1847: Extraits de deux lettres address´ees a M. Liouville. J. Math. Pures Appl., 12, 256–264.

condition changes along a given boundary from a Dirichlet condition to a Neumann condition and vice versa.

3- Heat Equation:

The development of using Green's function to solve the heat equation consists of two parts. In the nineteenth century a synthetical method was developed which replaces the actual distribution by sets of sources and sinks distributed over the boundaries and throughout the region under investigation. The twentieth century has been dominated by the use of transform methods.

Our tale begins with William Thomson (Lord Kelvin)19[19] (1824–1907) and his solution of the one-dimensional heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\quad 0<x<\infty,\qquad 0<t, \qquad (1.3.1)$$

with the boundary conditions

$$u(0,t)=\begin{cases}V, & 0<t<T\\ 0, & T<t,\end{cases}\quad \lim_{x\to\infty} u(x,t)\to 0,\qquad 0<t, \qquad (1.3.2)$$

and the initial conditions

$$u(x,0)=0,\qquad 0<x<\infty. \qquad (1.3.3)$$

Using Fourier integrals, Kelvin get the solution

$$u(x,t)=\frac{Vx}{\sqrt[2]{\pi}}\int_0^T \frac{dT}{(t-T)^{3/2}}exp\left[\frac{x^2}{4(t-T)}\right]. \qquad (1.3.4)$$

He also showed that the solution could also be obtained "synthetically" for small T if he introduced a source at $x = 0$ and a sink at $x = \alpha$ and took the limit as $\alpha \to 0$. Furthermore, reporting on some correspondence with George G. Stokes (1819–1903), he gave the solution in a form that we now call the superposition integral:

$$u(x,t)=\frac{x}{\sqrt[2]{\pi}}\int_0^t exp\left[-\frac{x^2}{4(t-\tau)}\right]\frac{f(T)}{t-\tau}d\tau, \qquad (1.3.5)$$

where $u(0, t) = f(t)$.

In 1887 Ernst William Hobson[20] (1856–1933) generalized Kelvin's synthetical method to two and three dimensions. In general, his effort must be considered a failure since it resulted in expressions that were difficult to interpret. For example, in another study,[21] Hobson redid

(19) Thomson, W., 1854/55: On the theory of the electric telegraph. Proc. R. Soc. London, 7, 382–399.

(20) Hobson, E. W., 1887: Synthetical solutions in the conduction of heat. Proc. London Math. Soc., Ser. 1, 19, 279–299.

(21) Hobson, E. W., 1888: On a radiation problem. Math. Proc. Cambridge Philos. Soc,.6,187,184

Kelvin's problem of one-dimensional heat conduction when the boundary condition at $x = 0$ changed to $u_x(0, t) = h[u(0, t) - f(t)]$, where h denotes the external conductivity. He found that

$$u(x,t) = \frac{h}{\sqrt[2]{\pi}} \int_0^\infty \int_0^\infty e^{-h\varsigma} \left\{ \exp\left[-\frac{(x+\varsigma-\xi)^2}{4t} \right] - \exp\left[-\frac{(x+\varsigma-\xi)^2}{4t} \right] \right\} \quad (1.3.6)$$

Hobson's solution fails if the initial temperature distribution is discontinuous In 1892, George H. Bryan[22] (1864–1928) improved the synthetical method. He cleverly wrote the solution as the sum of two parts: a source term (located at $x = \xi$) plus a homogeneous solution so that the total solution satisfies the boundary condition. Using this technique to redo Hobson's 1888 calculation, he found that

$$u(x,t) = \frac{1}{\sqrt[2]{\pi t}} \int_0^\infty \left\{ \exp\left[-\frac{(x-\xi)^2}{4t} \right] - \exp\left[-\frac{(x+\xi)^2}{4t} \right] \right\} f(\xi) d\xi$$
$$- \frac{h}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty e^{-h\varsigma} \exp\left[-\frac{(x+\varsigma-\xi)^2}{4t} \right] f(\xi) d\xi \, d\varsigma. \quad (1.3.7)$$

Later on, Sommerfeld[23] generalized Kelvin's results by considering substances that have different thermal properties. He considered two cases: (a) two semi-infinite domains with the interface at $x = 0$ and (b) two semi-infinite slabs separated a finite layer lying between $a < x < b$. In 1939 P. V. Solovieff[24] used mirror images to construct Green's functions for a n+1 dimensional heat equation when one or more boundaries are moving.

By the turn of the twentieth century, John Dougall[25] (1867–1960) introduced the concept of contour integration to find the mathematical description of how a sphere cools in a well-stirred liquid. Although Dougall's analysis made no use of Green's functions, it provided the necessary insight that allowed H. S. Carslaw to synthesize all of the ideas on how to apply Green's function to heat conduction problems. Carslaw's 1902 study[26] began by deriving how

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(22) Bryan, G. H., 1892: Note on a problem in the linear conduction of heat. Math. Proc. Cambridge Philos. Soc., 7, 246–248.

(23) Sommerfeld, A., 1894: Zur analytischen Theorie der W¨armeleitung. Math. Ann., 45, 277,362.

(24) Solovieff, P. V., 1939: Die Greensche Funktion der W¨ armeleitungsgleichung. Dokl. Acad. Sci. USSR, 23, 132–134; Solovieff, P. V., 1939: Fonctions de Green des ´equations paraboliques. Dokl. Acad. Sci. USSR, 24, 107–109.

(25) Dougall, J., 1901: Note on the application of complex integration to the equation of conduction of heat, with special reference to Dr. Peddie's problem. Proc. Edinburgh Math. Soc., Ser. 1, 19, 50–56.

(26) Carslaw, H. S., 1902: The use of Green's functions in the mathematical theory of the conduction of heat. Proc. Edinburgh Math. Soc., Ser. 1, 21, 40–64.

any conduction problem without sources can be expressed in terms of its Green's function, the initial condition and the solution's value along the boundary surrounding the domain of interest. The question then turned on the question of finding the Green's function for various geometries and boundary conditions. In the case of unbounded domains, he used the Green's functions given by the synthetical method. The remaining Green's functions were obtained using Dougall's method of contour integration. For example, to find the Green's function for linear heat flow over the interval (0, a), when the Green's function vanishes at both ends, Carslaw first introduced the Green's function

$$g_1(x,t|\xi,0)=\frac{1}{2\sqrt{\pi kt}}\left\{\exp\left[-\frac{(x-\xi)^2}{4t}\right]-\exp\left[-\frac{(x+\xi)^2}{4t}\right]\right\} \qquad (1.3.8)$$

The first term is the free-space Green's function given by Kelvin while the second term assures that $g_1(0, t|\xi, 0) = 0$. Carslaw then showed that this Green's function could be expressed by the contour integral

$$g_1(x,t|\xi,0)=\frac{1}{\pi i}\int_P e^{-kz^2t}\sin(x<z)\,e^{-izx>}dz, \qquad (1.3.9)$$

where P is the contour shown in Figure 1.3.2. On the right side, the contour must lie between $0 < \arg(z) < \pi/4$ as $|z| \rightarrow \infty$ while on the left side the contour lies between $3\pi/4 < \arg(z) < \pi$ as $|z| \rightarrow \infty$.

Next, Carslaw introduced the new Green's function

$$g_2(x,t|\xi,0)-\frac{1}{\pi i}\int_P e^{-kz^2t}\frac{\sin(x<z)\sin(x>z)}{\sin(az)}e^{-iaz}dz, \qquad (1.3.10)$$

Why did Carslaw create this new Green's function? Because a linear combination of $g_1(x, t|\xi, 0)$ and $g_2(x, t|\xi, 0)$ yields the Green's function

$$g(x,t|\xi,0)-\frac{1}{\pi i}\int_P e^{-kz^2t}\frac{\sin(x<z)\sin[a-x_>)z]}{\sin(az)}dz, \qquad (1.3.11)$$

which satisfies the boundary conditions $g(0, t|\xi, 0) = g(a, t|\xi, 0) = 0$. Furthermore, Carslaw was also able to show that this Green's function satisfied the initial condition and had the correct behavior at the singularity $x=\xi$., this Green's function could be rewritten in the convenient form of

$$g(x,t|\xi,\tau)=\frac{2}{a}\sum_{n=1}^{\infty} sin\left(\frac{\eta\pi x}{a}\right)\sin\left(\frac{\eta\pi\xi}{a}\right)\exp\left[\frac{k\eta^2\pi^2(t-\tau)}{a^2}\right]. \qquad (1.3.12)$$

The difficulty of this method is quite apparent: It is not easy to choose a complex representation for the Green's function that satisfies the boundary conditions, initial condition and the singular nature at the point of excitation. This difficulty became academic with the advent of Laplace transforms. In the mid-1920s, Gustav Doetsch (1892–1977) wrote a series of studies on heat conduction.[27] mainly, he considered the heat conduction problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < c, \qquad 0 < t, \qquad (1.3.13)$$

with the boundary conditions

$$u(0,t) = A(t), \quad u(c,t) = B(t), \quad 0 < t, \qquad (1.3.14)$$

and the initial condition

$$u(x,0) = \Phi(x), \qquad 0 < x < c. \qquad (1.3.15)$$

He showed that the solution u(x, t) could be expressed by the one-dimensional version of Equation 5.0.11 and the Green’s function.

$$g(x,t|\xi,0^+) = \frac{2}{c}\sum_{n=1}^{\infty} e^{-\frac{\eta^2\pi^2 t}{c^2}} sin\,(\eta\pi\xi/c)\sin\,(\eta\pi c/c). \qquad (1.3.16)$$

Furthermore, he showed that the Green’s function is symmetric. Finally, the limit of $c \to \infty$ yielded the Green’s functions on a semi-infinite domain found by Kelvin and Carslaw. The revolutionary aspect of Doetsch’s approach was his use of Laplace transforms.

In 1932, Sydney Goldstein[28] (1903–1989) showed how Laplace transforms could be used to solve many heat conduction problems whose derivation up to that time had been very cumbersome. Indeed he pointed out that many of the Green’s functions found by Carslaw’s contour integral method was in reality the “operational method".

Consider, for example, the one-dimensional heat conduction problem

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, \qquad 0 < t, \qquad (1.3.17)$$

Taking the Laplace transform of Equation 1.3.17, we have that

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(27) Bernstein, F., and G. Doetsch, 1925: Probleme aus der Theorie der W¨armeleitung.
I. Mitteilung. Eine neue Methode zur Integration partieller Differentialgleichungen. Der lineare W¨armeleiter mit verschwindender Anfangstemperatur. Math. Z., 22, 285–292; Doetsch, G., 1925: Probleme aus der Theorie der W¨ armeleitung. II. Mitteilung. Der lineare W¨armeleiter mit verschwindender Anfangstemperatur. Die allgemeinste L¨ osung und die Frage der Eindeutigkeit. Math. Z., 22, 293–306; Doetsch, G., 1925: Probleme aus der Theorie der W¨armeleitung. III. Mitteilung. Der lineare W¨armeleiter mit beliebiger Anfangstemperatur. Die zeitliche Fortsetzung des W¨ armezustandes. Math. Z., 25, 608–626; Bernstein, F., and G. Doetsch, 1927: Probleme aus der Theorie der W¨armeleitung. IV. Mitteilung. Die r¨aumliche Fortsetzung des Temperaturablaufs (Bolometerproblem). Math. Z., 26, 89–98.

(28) Goldstein, S., 1932: Some two-dimensional diffusion problems with circular symmetry. Proc. London Math. Soc., Ser. 2, 34, 51–88.

$$\frac{d^2U}{dx^2} = q^2U, \qquad 0 < x < \infty, \qquad (1.3.18)$$

where s = a2q2. The solution to Equation 1.3.18 is

$$U(x,s) = Ae^{-qx}, \qquad (1.3.19)$$

where we have discarded the exponentially growing solution as $x \to \infty$. Now

$$\int_0^U (x,s)dx = 2A(1 - e^{-qx})/q. \qquad (1.3.20)$$

To find the Green's function, we must choose A so that it represents an instantaneous plane source of heat from $x = -\infty$ to $x = \infty$ in the limit of $t \to 0$ In that case, the left side of Equation 1.3.20 equals one as $q \to \infty$, or $2A = q$ Therefore

$$G(x,S|1,0) = \frac{1}{2}qe^{-qx} \qquad (1.3.21)$$

Taking the inverse of G(x, s|0, 0), we have that

$$g(x,t|0,0) = \frac{1}{2\sqrt{\pi a^2 t}} \exp\left(-\frac{r^2}{4a^2t}\right) \qquad (1.3.22)$$

the same result that Kelvin found. Goldstein used similar methods to find the Green's function for an axisymmetric problem in the plane and outside of a cylinder.

During the 1930s and 1940s several authors found the Green's functions for cylindrical and spherical geometries by using Byran's method of writing the Green's function as a sum of a free-space Green's function plus a homogeneous solution which satisfies the boundary conditions. They obtained the homogeneous solution using Laplace transforms. For example,(29) Lowan redid Bryan's original problem of finding the Green's function in a semi-infinite planar solid that is radiating at the $x = 0$ face. Later, Lowan applied this technique to heat conduction in cylindrical(30) and spherical(31) coordinates.

During this same period Carslaw and Jaeger also found Green's functions using Laplace transforms. In their earliest study(32) they found the Green's function for the region outside of a

(29) Lowan, A. N., 1937: On the operational determination of Green's functions in the theory of heat conduction. Philos. Mag., Ser. 7, 24, 62–70.

(30) Lowan, A. N., 1938: On the operational determination of two dimensional Green's function in the theory of heat conduction. Bull. Amer. Math. Soc., 44, 125–133.

(31) Lowan, A. N., 1939: On Green's functions in the theory of heat conduction in spherical coordinates. Bull. Amer. Math. Soc., 45, 310–315 and 951–952.

(32) Carslaw, H. S., and J. C. Jaeger, 1939: On Green's functions in the theory of heat conduction. Bull. Amer. Math. Soc., 45, 407–413.

cylinder using both the Laplace transforms and the contour method. In a subsequent study Carslaw[33] found the Green's function for heat conduction in two semi-infinite solids of different materials with a common boundary at $x = 0$ as well as the case when the two semiinfinite solids are separated by a third solid of thickness 2_a. In the case of three dimension problems, they[34] wrote the Green's function as a sum of the free-space Green's function plus a homogeneous solution of the heat equation. They then used Laplace transforms to find the homogeneous solution. Finally Carslaw and Jaeger[35]applied Bryan's technique to cylindrical problems where they used line sources for the free-space Green's functions.

4-Helmholtz's Equation:

A partial differential equation which is quite similar to Laplace's equation is Helmholtz's equation. It arises in the study of forced (steady-state) vibrations governed by the wave equation; the most famous application is the diffraction of acoustic and visible light waves

The history of Green's function involving Helmholtz's equation begins with the theoretical work of Hermann von Helmholtz (1821–1894) during his study of acoustics.[36] He showed that the free-space Green's function is $g(x, y, z|\xi, \eta, \zeta)=\cos(k_0 r)/r$, where $r^2= (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ Helmholtz used this Green's function to express the solution of $\nabla^2 u+k_0^2 u=0$ in a region R with the boundary S which gives the solution and its derivative on the boundary.

F. Pockels (1865–1913) epic book[37] on Helmholtz's equation summarized our knowledge of Green's function at the end of the nineteenth century. In Part IV (Bestimmung de Functionen u aus gegebenen Randwerthen und verwandten Bedingungen) he reviewed the Dirichlet principle and showed how Green's functions could be used to solve this problem. Next Pockels discussed the expansion of Green's function in term of eigenfunctions. For example, he gave the Green's function within a circle in terms of Fourier series.

In the early twentieth century several major studies appeared on the Green's functions

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(33) Carslaw, H. S., 1940: A simple application of the Laplace transformation. Philos. Mag., Ser. 7, 30, 414–417.

(34) arslaw, H. S., and J. C. Jaeger, 1940: The determination of Green's function for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. J. London Math. Soc., 15, 273–281.

(35) Carslaw, H. S., and J. C. Jaeger, 1941: The determination of Green's function foline sources for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. Philos. Mag., Ser. 7, 31, 204–208.r

(36) Helmholtz, H., 1860: Theorie der Luftschwingungen in R¨ohren mit offenen Enden. J. Reine Angew. Math., 57, 1–72.

(37) Pockels, F., 1891: Uber die partielle Di ¨ fferentialgleichung Δu + k2u = 0 und deren Auftreten in der mathematischer Physik. Leipzig, Teubner, 339 pp.

for Helmholtz's equation. The first study[38] was done by A. Sommerfeld (1868–1951). It consisted of two parts: Green's function for a bounded and unbounded region. For a finite domain he showed that the Green's func tion can be expressed by the eigenfunction expansion:

$$g = \sum_{m} \frac{{}^{u}m(O)\ {}^{u}m(P)}{k_0^2 - k_,^2} \qquad (1.4.1)$$

where ${}^{u}m$(O) and ${}^{u}m$(P) are the *m*th eigenfunction to the eigenvalue problem$\nabla^{2u}m+k_3^2\,{}^{u}m$=0 with ${}^{u}m$=0 along the boundary and point *O* denotes a general point within the domain while the point *P* is the location of the singularity. Sommerfeld proved that this expansion satisfies the partial differential equation, *g*=0 at the boundary, is indefinite when point *P* and point *O* are collocated, and satisfies reciprocity. As examples, he gave freespace Green's functions in two and three dimensions as well as for a circular membrane with a fixed boundary and a three-dimensional parallelepiped with Neumann boundary conditions. In two subsequent papers H. S. Carslaw39[39] extended Sommerfeld's results for a wide variety of three dimensional spaces involving cylindrical and spherical coordinates.

Turning to unlimited domains, Sommerfeld gave the free-space Green's function in two and three dimensions. More importantly, he derived his famous "radiation condition" that required outwardly propagating waves from physical considerations.40[40]

By 1950 Green's functions for Helmholtz's equation were used to find the wave motions due to flow over a mountain41[41] and in acoustics.[42]

5-Wave Equation:

Soon after the publication of Green's essay, Green's functions were used to solve the wave equation. In 1860 Bernhard Riemann[43] (1826–1866) applied the method of Green's

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[38] Sommerfeld, A., 1912: Die Greensche Funktion der Schwingungsgleichung. Jahresber. Deutsch. Math.-Verein., 21, 309–353.

[39] Carslaw, H. S., 1912: Integral equations and the determination of Green's functions in the theory of potential. Proc. Edinburgh Math. Soc., Ser. 1, 31, 71–89; Carslaw, H. S., 1914: The Green's function for the equation ∇2u + k2u = 0. Proc. London Math. Soc,. Ser. 2, 15, 236–257

[40] See Schot, S. M., 1992: Eighty years of Sommerfeld's radiation condition. Hist. Math401,385,19.,

[41] Lyra, G., 1943: Theorie der station¨ aren Leewellenstr¨omung in freier Atmosph¨are. Zeit. Angew. Math. Mech., 23, 1–28.

[42] Foldy, L. L., and H. Primakoff, 1945: A general theory of passive linear electroacoustic transducers and the electroacoustic reciprocity theorem. I. J. Acoust. Soc. Am., 17, 109,120.

[43] Riemann, B., 1860: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Abh. d. K¨on. Ges. der Wiss. zu G¨ottingen, 8, 43–65. An English translation appears in Johnson, J. N., and R. Ch´eret, 1998: Classic Papers in Shock Compression Science. Springer-Verlag, 524 pp.

functions to integrate the hyperbolic equation that describes the propagation of sound waves.[44]

For a linear hyperbolic equation of second order in two independent variables

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} + a^2 \frac{\partial u}{\partial t} - 2b\frac{\partial u}{\partial t} + cu = 0, \qquad (1.5.1)$$

where x and y are chosen so that the two families of characteristic are $x \pm y$ = constant and a, b and c are functions only of x and y, the solution[45] at the point P with coordinates (ξ, η) is

$$u(\xi,\eta) = \frac{1}{2}(uG)\Big|_A + \frac{1}{2}(uG)\Big|_B$$

$$+\frac{1}{2}\int_{AB}\big(Gu_y - uG_y + 2buG\big)dx + (Gu_x - uv_x + 2a\,uG)dy. \qquad (1.5.2)$$

Here G denotes the Riemann-Green function and is given by the adjoint equation

$$\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial x^2} - 2\frac{\partial (aG)}{\partial x} + 2\frac{\partial (bG)}{\partial y} + cG = 0, \qquad (1.5.3)$$

such that

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = (a+b)G \qquad on\ y - x = \eta - \xi, \qquad (1.5.4)$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = (a-b)G \qquad on\ y + x = \eta + \xi, \qquad (1.5.5)$$

and $G(\xi,\eta) = 1$. The values of u and its first derivative are specified along the arc AB which is chosen so that no characteristic cuts it more than at one point the arcs PA and PB are characteristics. Although there are several methods for finding Riemmann-Green functions,[46] actually finding one is very difficult. The greatest success with this technique involved the equation of telegraphy.[47] The next important development of Green's functions for the wave

[44] Mackie, A. G., 1964/65: Green's function and Riemann's method. Proc. Edinburgh Math. Soc., Ser. 2, 14, 293–302.

[45] For the derivation, see Section 73 in Webster, A. G., 1966: Partial Differential Equations of Mathematical Physics. Dover, 446 pp.

[46] See Copson, E. T., 1958: On the Riemann-Green function. Arch. Rat. Mech. Anal,. 1,348,324.

[47] Picard, E., 1894: Sur l'´ ´ equation aux d´eriv´ees partielles qui se recontre dans la th´eorie de la propagation de l'´electricit´e. Acad. Sci., Compt. Rend., 118, 16–17; Bois-Reymond, P. du, 1889: Uber lineare partielle Di ¨ fferentialgleichungen zweiter Ordnung. J. Reine Angew. Math., 104, 241–301; Voigt, W., 1899: Ueber die Aenderung der Schwingungsform des Lichtes beim Fortschreiten in einem dispergirenden oder absorbirenden Mittel. Ann. Phys;,.603,304,589 Gray, M. C., 1923: The equation of telegraphy. Proc. Edinburgh Math. Soc,. Ser. 2, 42, 14–28; Rademacker, H., and R. Iglisch, 1961: Randwertprobleme der partiellen Differentialgleichungen

equation lies with Gustav Robert Kirchhoff[48] (1824–1887), who used it during his study of the three-dimensional wave equation. Starting with Green's second formula, he was able to show that the three-dimensional Green's function is

$$g(x,y,z,t|\xi,\eta,\varsigma,T) = \frac{\delta(t-T-R/c)}{4\pi R}, \qquad (1.5.6)$$

where $R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\varsigma)^2}$ (modern terminology). Although he did not name his solution a Green's function,[49] he clearly grasped the concept that this solution involved a function that we now name the Dirac delta function He used this solution to derive his famous Kirchhoff's theorem, which is the mathematical expression for Huygen's principle: energy always propagates out to infinity.

The early twentieth century saw the development of Laplace transforms to solve the wave equation in addition to the previous known method of Fourier transforms.[50] In 1914 T. J. I'A. Bromwich (1875–1929) showed how Laplace transforms51[51] can be used to solve the wave equation by eliminating the temporal dependence, leaving a boundary-value problem. Interestly he then solved this boundary-value problem using Green's functions. Then, unknowingly he found as an example the Green's function for the one-dimensional wave equation with fixed ends. A. N. Lowan applied Bromwich's idea to finding the wave motions within a wedge[52] of infinite radius and an infinite solid[53] which is exterior to a cylinder or

zweiter Ordnung, 779–828 in Frank, Ph., and R. von Mises, 1961: Die Differential- und Integralgleichungen der Mechanik und Physik. I. Mathematischer Teil. Dover, 916 pp.; Section 74 in Webster, A. G., 1966: Partial Differential Equation of Mathematical Physics. Dover, 446 pp.; Wahlberg, C., 1977: Riemann's function for a Klein-Gordon equation with a non-constant coefficient. J. Phys., Ser. A, 10, 867–878; Asfar, O. R., 1990: Riemann-Green function solution of transient electromagnetic plane waves in lossy media. IEEE Trans. Electromagn. Compat., EMC-32, 228–231.

[48] Kirchhoff, G., 1882: Zur Theorie der Lichtstrahlen. Sitzber. K. Preuss. Akad. Wiss. Berlin, 641–669; reprinted a year later in Ann. Phys. Chem., Neue Folge, 18, 663–695.

[49] This appears to have been done by Gutzmer, A., 1895: Uber den analytischen Aus -druck des Huygens'schen Princips. J. Reine Angew. Math., 114, 333–337.

[50] Poincar´e, H., 1893: Sur la propagation de l'´electricit´e. Acad. Sci., Compt. Rend,. 117 ,1032,1027 ;Webster, A. G., 1966: Partial Differential Equation of Mathematical Physics. Dover, Section 46.

[51] Bromwich, T. J. I'A., 1914: Normal coordinates in dynamical systems. Proc. London Math. Soc., Ser. 2, 15, 401–448.

[52] Lowan, A. N., 1941: On the problem of wave-motion for the wedge of an angle. Philos. Mag., Ser.7, 31, 373–381

[53] Lowan, A. N., 1939: On wave motion in an infinite solid bounded internally by a cylinder or a sphere. Bull. Amer. Math. Soc., 45, 316–325.

sphere.

One difficulty of finding the Green's function for the wave equation lies in its definition. Around 1950 A. G. Walters wrote a series of study[54] (that have been essentially forgotten) on finding the Green's function of transient partial differential equations. In the case of the wave equation, his Green's vibrational function g(P, P_1, $t-\tau$) satisfied the wave equation.

$$\frac{\partial^2 g}{\partial t^2} = c^2 D(g), \qquad (1.5.7)$$

where D(.) is a differential operator in one or more dimensions, and satisfies the integral conditions

$$\lim_{T \to t} \int g(P, P_1, t - T) dV = 0, \qquad (1.5.8)$$

and

$$\lim_{T \to t} \int gt(P, P_1, t - T) dV = 1, \qquad (1.5.9)$$

where V is any region enclosed by the boundaries and contains the point P_1, which is defined below. He then showed how the solution to the wave equation can be expressed in terms of volume integrals involving the initial conditions, boundary conditions, and an source terms. To compute g(P, P_1, t−τ), Walters proved that the Green's function is the inverse Laplace transform of Φ (P, P_1, s^2/c^2), where Φ satisfies the boundary-value problem

$$D(\Phi) = \frac{s^2}{c^2} \Phi. \qquad (1.5.10)$$

Note that Φ satisfies Equation 1.5.10 except for a singular point located at P_1

Our modern definition of the Green's function for the wave equation as the response of this equation to impulse forcing appears to originate with H. J. Bhabha[55] (1909–1966) in his study of the meson field of neutrons, he used transform methods to find the Green's function for the Klein-Gordon equation

Van der Pol and Bremmer were the first to introduce the general community to the concept that the Green's function[56] of the wave equation is a particular solution to the wave

[54] Walters, A. G., 1949: The solution of some transient differential equations by means of Green's functions. Proc. Cambridge Philos. Soc., 45, 69–80; Walters, A. G., 1951: On the propagation of disturbances from moving sources. Proc. Cambridge Philos. Soc., 47,126-109.

[55] Bhabha, H. J., 1939: Classical theory of mesons. Proc. R. Soc. London, Ser. A, 172, 409-384.

[56] Van der Pol, B., and H. Bremmer, 1964: Operational Calculus Based on the Two-Sided Laplace Transform. Cambridge, 415 pp.

equation when it is forced by a point source both in space and time. They then derived the Green’s function for the ndimensional wave equation as well as the three-dimensional wave equation with dispersion. Shortly after Van der Pol’s study, P. M. Morse and H. Feshbach further developed the theory of Green’s functions as it applies to the wave equation. From Van der Pol’s definition, they obtained the reciprocity relation and the free-space Green’s function in one, two, and three dimensions.(57) Significantly Morse and Feshbach did not use transform methods but derived their results from heuristic arguments and repeated integration.

The use of transform methods to find Green’s function for the wave equation rapidly occurred after the publication of Van der Pol’s study. For example in Friedlander’s examination(58) of pulses by a circular cylinder, he found the approximate Green’s function for the two-dimensional wave equation exterior to a cylinder of radius 1 where $g_r(1\ \theta, t|\rho, \theta\ \$, \tau) = 0$. In particular, Laplace transforms were used to eliminate time and Fourier series were employed to give the θ dependence.

An important class of wave propagation involves the diffraction of a direct wave by an infinitesimally thin barrier along the *x*-axis. In 1935 L. Cagniard used Laplace transforms to find the diffraction of a step function direction by a half-plane.(59) The Green’s function follows by simply taking the time derivative of Cagnaird’s solution.(60)

The Green’s function for the corresponding two-dimensional problem is more difficult. R. D. Turner found the earliest representation using Laplace transforms.(61) G. Schouten(62) has given a closed form solution for the Green’s function.

6-Ordinary Differential Equations:

The application of Green’s functions to ordinary differential equations began in 1894. Noting the use of Green’s functions in solving the two and three dimensional Poisson equation,

(57) Morse, P. M., and H. Feshbach, 1953: Methods of Theoretical Physics. Part I: Chapters 1 to 8. McGraw-Hill, 997 pp.

(58) Friedlander, F. G., 1954: Diffraction of pulses by a circular cylinder. Commun. Pure Appl. Math., 7, 705–732.

(59) Cagniard, L., 1935: Diffraction d’une onde progressive par un ´ecran en forme de demiplan. J. Phys. Radium, Ser. 7, 6, 310–318; Cagniard, L., 1935: Diffraction d’une onde harmonique par un ´ecran en forme de demi-plan. J. Phys. Radium, Ser. 7, 6, 369–372.

(60) Schouten, G., 1999: Two-dimensional effects in the edge sound of vortices and dipoles. J. Acoust. Soc. Am., 106, 3167–3177.

(61) Turner, R. D., 1956: The diffraction of a cylindrical pulse by a half-plane. Q. Appl. Math., 14, 63–73.

(62) Schouten, op. cit., p. 3170.

H. Burkhardt[63] (1861–1914) asked whether they could be used to solve

$$\frac{d^2y}{dx^2} = f(x), a < x < b. \qquad (1.6.1)$$

He showed that the solution to this problem can be written

$$y(x) = -\int_a^x \frac{(b-x)(\xi - a}{b-a} f(\xi)d\xi - \int_x^b \frac{(b-\xi)(x-a)}{b-a} f(\xi)d\xi , \qquad (1.6.2)$$

which he wrote

$$y(x) = -\int_a^b g(x|\xi)f(\xi)d\xi, \qquad (1.6.3)$$

Where

$$g(x|\xi) = \frac{(b-x_>)(x_< - a)}{b-a}. \qquad (1.6.4)$$

Burkhardt's Green's function $g(x|\xi)$ enjoyed the classic properties:

- The Green's function satisfies the differential equation $g''= 0$.
- The Green's function is finite and continuous on the interval (a, b) except for $x = \xi$
- The first derivative of $g(x|\xi)$ is continuous except at $x=\xi$ where it possesses two distinct values that differ by 1
- $g(a|\xi) = g(b|\xi) = 0$-
- The Green's function has the symmetry property: $g(x|\xi) = g(\xi|x)$.

In 1903 M. Bôcher[64] (1867–1918) gave the properties of the Green's function for homogeneous, linear, n-th order ordinary differential equation

$$\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_{ny} = 0, \qquad a \le x \ge b.$$

(1.6.5)

where a_1, a_2, . . . , an are real functions of the real variable x and the coefficients of the n linearly independent boundary conditions are constants. William M. Why burn (1901–1972) would list[65] the properties of the Green's function for the general self-adjoint linear, second-order homogeneous differential equation

(63) Burkhardt, M. H., 1894: Sur les fonctions de Green relatives a un domaine d'une dimension. Bull. Soc. Math., 22, 71–75.

(64) Bôcher, M., 1901: Green's function in space of one dimension. Bull. Amer. Math. Soc., Ser. 2, 7, 297–299.

(65) Whyburn, W. M., 1924: An extension of the definition of the Green's function in one dimension. Ann. Math., Ser. 2, 26, 125–130.

$$\frac{d}{dx}\left[p(x,\lambda)\frac{dy}{dx}\right]-q(x,\lambda)y=0,\quad a<x<b, \tag{1.6.6}$$

with self-adjoint boundary conditions. The jump in the first derivative at the point of excitation $x = \xi$ now becomes $1/p(x,\lambda)$

Burkhardt also found the Green's function for Equation 1.6.1 when the boundary condition reads

$$y(a)+Xy'(a)=0,\quad yb+Xy'(b)=0. \tag{1.6.7}$$

In that case

$$g(x|\xi)=\frac{(b+x-x_{>})(x_{<}-a+x)}{b-a+2_X} \tag{1.6.8}$$

He noted special difficulties when $\chi = \infty$ and $\chi = (a - b)/2$. In the first case the boundary condition becomes $g'(a|\xi) = g'(b|\xi) = 0$. For the solution to exist, $\int_a^b f(x)dx=0$. In that case

$$y(x)=\int_a^x f(\xi)(x-\xi)d\xi+\lambda, \tag{1.6.9}$$

where λ is arbitrary. In the second case, the solution exists if

$$\int_a^b\left(x-\frac{a+b}{2}\right)f(x)dx=0. \tag{1.6.10}$$

If this condition holds, then

$$y(x)=\int_a^x f(\xi)(x-\xi)d\xi+\lambda,\left(x-\frac{a+b}{2}\right). \tag{1.6.11}$$

All of these results were merely stated, not proved. In 1911 M. Bôcher(66) provided the mathematical justification for these results Ince,(67) in his great treatise on ordinary differential equations, introduced these results to the general community. In his Section 11.1 he proved that the Green's function for Equation 1.6.5 exists and is unique. Furthermore he showed that the Green function is symmetrical $g(x|\xi) = g(\xi|x)$ if the ordinary differential equation is self-adjoint.

$$y(x)=\int_a^b g(x|\xi)f(\xi)d\xi \tag{1.6.12}$$

is the solution to the non-homogeneous equation

$$\frac{d^n u}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_n y=f(x),\qquad a\le x\le b. \tag{1.6.13}$$

(66) Bˆocher, M., 1911/12: Boundary problems and Green's functions for linear differential and difference equations. Ann. Math., Ser. 2, 13, 71–88.

(67) Ince, E. L., 1956: Ordinary Differential Equations. Dover, 558 pp.

an important concept in the theoretical development of Green's functions as they apply to ordinary differential equations involves the adjoint. In 1908 G. D. Birkhoff[68] formulated the Green's function in terms of the adjoint differential equations and the eigenvalues and eigenfunctions of the differential equations. He expressed the Green's function as a ratio of determinants in terms of linearly independent solutions to the differential equation and the boundary conditions. Then, in 1909 E. Bounitzky[69] extended Birkhoff results to systems of first-order differential equations.

Of all of the possible ordinary differential equations that possess a Green's function, several are of fundamental importance because these arise in important physical problems. One of these is

$$\frac{d^2u}{dx^2} + [\ell(x) + \lambda^2]u = 0, \qquad 0 < x < 1, \qquad (1.6.14)$$

with $u(0) = u(1) = 0$. In 1911 Hilb[70] (1882–1929) wrote down the Green's functions in terms of linearly homogeneous solutions to Equation 1.6.14. Furthermore, he found the bilinear representation of the Green's function

$$g(x|\xi) = \sum_j \frac{\ell_j(\xi)\ell_j(x)}{\lambda_j^2 - \lambda^2}, \qquad (1.6.15)$$

where λ_j and $\ell_j(x)$ are the eigenvalues and eigenfunctions, respectively, of the boundary-value problem

Another differential equation that arises from physics is the simple harmonic oscillator. In this case, the Green's function is governed by

$$g'' + k^2 g = \delta(x - \xi). \qquad (1.6.16)$$

In 1910 A. Sommerfeld (1868–1951) examined the solution[71] to Equation1.6.16 over a finite interval $0<x<L$ when the boundary conditions are $g(0|\xi) = g(L|\xi) = 0$. He expressed the Green's function by the expansion

(68) Birkhoff, G. D., 1908: Boundary value and expansion problems of ordinary linear differential equations. Trans. Amer. Math. Soc., 9, 373–395.

(69) Bounitzky, E., 1909: Sur la fonction de Green des ´equations diff´erentielles lin´eaires ordinaires. J. Math. Pures Appl., Ser. 6, 5, 65–126.

(70) Hilb, E., 1911: Uber Reihenentwicklungen nach den Eigenfunktionen linearer Diffe¨ - rentialgleichungen 2ter Ordnung. Math. Ann., 71, 76–87.

(71) Sommerfeld, A., 1910: Die Greensche Funktion der Schwingungsgleichungen f¨ ur ein beliebiges Gebiet. Phys. Z., 11, 1057–1066.

$$g(x|\xi) = \sum_{m=1}^{\infty} \frac{u_m(x)u_m(\xi)}{k^2 - k_m^2}, \qquad (1.6.17)$$

where km and $u_m(x)$ are the eigenvalues and eigenfunctions of the SturmLiouville problem: $u_m'' + k^2u_m = 0$ with $u_m(0) = u_m(L) = 0$.

The bilinear expansion found by Sommerfeld for the harmonic oscillator is a simple example of the general Sturm-Liouville problem:

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda u, \quad 0 < x, < 1, \qquad (1.6.18)$$

with $u(0) = u(1) = 0$. In 1911, Adolf Kneser(72) (1862–1930) showed how the symmetric Green's function for this problem could be formulated as the homogeneous integral equation

$$u(\xi) = \lambda \int_0^1 g(x|\xi)u(x)dx. \qquad (1.6.19)$$

He showed that this integral equation lead directly to the general bilinear formula

$$g(x|\xi) = \sum_{m=1}^{\infty} \frac{\psi_m(x)\psi_m(\xi)}{\lambda_m} \qquad (1.6.20)$$

where k_m and $\psi_m(x)$ are the mth eigenvalue and eigenfunction, respectively of Equation 1.6.18.

In the following year Sommerfeld considered the case when the interval is infinite $-\infty < x < \infty$. He showed(73) that the Green's function in this case is

$$g(x|\xi) = \frac{1}{2ik}e^{\pm ik(x-\xi)}, \qquad (1.6.21)$$

where the positive sign holds if $x > \xi$ and the negative sign holds when $x < \xi$. To find this solution, Sommerfeld(74) introduced his famous "radiation condition" that requires that energy always propagates out to infinity. Another important ordinary differential equation whose Green's function was studied in the early twentieth century is the one that governs the deflection of a beam(75) with clamped ends:

$$g^{iv} = \delta(x -), \qquad 0 < x < L, \qquad (1.6.22)$$

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(72) Kneser, op. cit.

(73) Sommerfeld, A., 1912: Die Greensche Funktion der Schwingungsgleichung. Jahrber. Deutsch. Math.-Verein., 21, 309–353.

(74) Schot, S. M., 1992: Eighty years of Sommerfeld's radiation condition. Hist. Math,. 19,401,385.

(75) See Section 1.22 and 1.23 in Bateman, H., 1959: Partial Differential Equations of Mathematical Physics. Cambridge, 522 pp. See also Von Mises, R., Ph. Frank, H. Weber, and B. Riemann, 1925: Die Differential- und Integralgleichungen der Mechanik und Physik. Vol I. Braunschweig, F. Vieweg, 687 pp.

with the boundary conditions

$$g(0|\xi) = g(L|\xi) = g''(0|\xi) = g''(L|\xi) = 0. \qquad (1.6.23)$$

That

$$g(x|\xi) = -\frac{x_<(x_> - L)}{6L}(x_<^2 + x_>^2 - 2Lx_>) \qquad (1.6.24)$$

A quick check shows that Equation 1.6.24 satisfies the ordinary differential equation and boundary conditions. Furthermore, $g'(\xi|\xi)$ and $g'(\xi|\xi)$ is continuous while

$$\frac{d^3g}{dx^3}\bigg|_{\xi^-}^{\xi^+} = -1. \qquad (1.6.25)$$

In 1914 A. Kneser(76) published a paper on the integral equation that expresses the vibrations of a string where the right boundary is attached to a mass and a spring. His analysis involved solving the Green's function problem (in modern notation) of

$$g'' + \lambda^2 g = -\delta(x - \xi), \qquad 0 < x, \xi < L, \qquad (1.6.26)$$

With

$$g(0|\xi) = g'(L|\xi) + (p - q\lambda^2)g(L|\xi) = 0, \qquad (1.6.27)$$

where $\lambda \neq \lambda_n$, the eigenvalue of the system, and $q > 0$. This problem is of particular interest because λ appears in the differential equation and the boundary condition. He found that the corresponding Green's function is

$$g(x|\xi) = \frac{\{(q\lambda^2 - p)\sin[\lambda(L - x_>)] - \lambda_{cos}[\lambda(L - x_>)]\}\sin(\lambda x_<)}{\lambda(q\lambda^2 - p)\sin(\lambda L) - \lambda^2\cos(\lambda L)}$$

(1.6.28)

He also found the bilinear expansion for this Green's function

the differential equations were incompatible there are no nonvanishing solutions which satisfy both the homogeneous differential equation and the boundary conditions. However, as early as 1904, Hilbert(77) gave simple examples of compatible differential equations for linear second-order, boundary-value problems of the form

$$L[u] = [p(t)u']' - q(t)u = 0. \qquad (1.6.29)$$

He then constructed a generalized Green's function (Greensche Funktionen im erweiterten Sinne) to treat such cases. In 1909, Westfall(78) proved Hilbert's results. Almost two

(76) Kneser, A., 1914: Belastete Integralgleichungen. Rend. Circ. Matem. Palermo, 37, 197-169.

(77) See pp. 44–45 in Hilbert, D., 1912: Grundz¨ uge einer allgemeiner Theorie der linearen Integralgleichungen. Leipzig, B. G. Teubner, 312 pp.

(78) Westfall, W. D. A., 1909: Existence of the generalized Green's function. Ann. Math,. Ser. 2, 10, 177–180.

decades later, Elliott[79] generalized Hilbert's results by finding the generalized Green's function for a His results are not limited to self-adjoint systems. *n*th order differential system,

Since these early studies, the study of generalized Green's functions has languished. Loud[80]used regular differential operator theory to explain generalized Green's functions and developed techniques to find them. In 1977 Locker[81] explored the properties of generalized Green's functions.

Conclusion:

There are many ways to solve differential equations, where at first the green function theory seems finite and limited to certain equations but not all linear operators can be used while these types of equations are always well known in physics so green function is expensive to understand Physico-mechanical systems and applied mathematics.

[79] Elliott, W. W., 1928: Generalized Green's function for compatible differential systems. Amer. J. Math., 50, 243–258; Elliott, W. W., 1929: Green's function for differential systems containing a parameter. Amer. J. Math., 51, 397–416.

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