



A Well-Posed and Dissipative Dynamical System for the Logarithmic Schrödinger Equation

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Abstract

This paper investigates the global dynamics and long-term behavior of solutions to the weakly damped logarithmic Schrödinger equation (LogSE) subject to a stationary external force in $H^1(\mathbb{R})$. By establishing suitable energy functionals, $J(u)$ and $F(u)$, and employing refined energy estimates alongside Gronwall-type differential inequalities, we rigorously establish the global existence and uniqueness of solutions in $H^1(\mathbb{R})$. Furthermore, we prove the existence of a bounded absorbing set $\mathcal{B}_0$ in $H^1(\mathbb{R})$, thereby confirming the global dissipation of the generated dynamical semi-group $S(t)$. Explicit estimates for the entry time t_0 into the absorbing ball and its corresponding radius M_2 are explicitly derived to account for the non-Lipschitz logarithmic singularity at the origin.

Keywords: Logarithmic Schrödinger equation; Global dissipation; Bounded absorbing set; Energy estimates; $H^1(\mathbb{R})$ global solutions.

1 Introduction

As an extension of earlier works dedicated to the functional analysis of the nonlinear logarithmic Schrödinger equation (LogSE) [33, 2], this paper focuses exclusively on the long-term asymptotic dynamics of the system [7, 19]. While global existence and uniqueness of solutions (Hadamard well-posedness) were established in a detailed preliminary study [10, 11, 20], we shift our attention here toward the global dissipative properties induced by damping mechanisms [4, 22, 35]. The primary objective of this article is to mathematically prove the existence of a bounded absorbing set in the Sobolev space $H^1(\mathbb{R})$ [16, 17, 27, 1]. The equation under consideration incorporates a linear damping mechanism as well as a stationary external force. It is formulated as follows:

$$iu_t + \Delta u + u \ln(|u|^2) + i\gamma u = f. \quad (1)$$

Here, the unknown is the complex-valued function $u : \mathbb{R}_x \times \mathbb{R}_t \rightarrow \mathbb{C}$, denoted by $u(t) = u(t, \cdot) \in H^1(\mathbb{R})$. The parameter $\gamma > 0$ represents the damping coefficient, which drives energy dissipation, while the function $f \in L^2(\mathbb{R})$ models the external force, assumed to be time-independent.

From a physical perspective, this equation plays a vital role in modeling dissipative quantum mechanics phenomena [25, 28], supersymmetric quantum mechanics [5, 8, 30], and solitary wave propagation in nonlinear optics [15, 23, 24]. Similar asymptotic behaviors and stationary profile techniques have been extensively analyzed in related quasi-linear and Schrödinger-Poisson models [12, 13, 14]. On a purely mathematical level, the logarithmic restoring force $u \ln(|u|^2)$ introduces a major technical challenge: it features a non-Lipschitz singularity at the origin (where $u = 0$). This singularity breaks standard energy estimates

and requires a careful analytical treatment to ensure control over the trajectories at infinity [18, 34, 36].

2 Defining a Semi-group

2.1 The Cauchy Problem

To ensure that the dynamical semi-group associated with our system is well-defined, analyzing the Cauchy problem is an indispensable preliminary step. This problem has been fully established in our complementary work [3]. Consequently, to avoid any redundancy, we refer the reader to that work for detailed proofs.

2.2 Dissipativity of the Semi-group

In this section, we focus on studying the global dissipative properties of the semi-group. In particular, we will establish the existence of a bounded absorbing set $\mathcal{B}_0$ in $H^1(\mathbb{R})$.

We begin by proving the existence of a bounded absorbing set in $L^2(\mathbb{R})$, which is what implicitly shows that $S(t)$ is well-defined for all positive times.

Lemma 1. Let u be a solution to (1). Then, we have

$$\|u\|_{L^2}^2 \leq e^{-\gamma t} \|u_0\|_{L^2}^2 + \frac{\|f\|_{L^2}^2}{\gamma^2} (1 - e^{-\gamma t}).$$

Proof. Multiplying equation (1) by $\bar{u}$ and taking the imaginary part of the resulting system, it follows the classical properties of Schrödinger operators quantity, it follows from the classical properties of Schrödinger operators [6, 9, 21], as well as spectral bounds for linear operator perturbations [26] and general partial differential systems [29] that:

$$(iu_t, \bar{u}) + (\Delta u, \bar{u}) + (i\gamma u, \bar{u}) + (u \ln |u|^2, \bar{u}) = (f, \bar{u}).$$

We then make use of the following identities

$$(iu_t, \bar{u}) = \operatorname{Re} \int u_t \bar{u} = \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2,$$

$$(\Delta u, \bar{u}) = \operatorname{Im} \int \Delta u \bar{u} = -\operatorname{Im} \int |\nabla u|^2 = 0,$$

$$(i\gamma u, \bar{u}) = \operatorname{Im} \int i\gamma u \bar{u} = \gamma \operatorname{Re} \int |u|^2 = \gamma \|u\|_{L^2}^2,$$

$$(u \ln |u|^2, \bar{u}) = \operatorname{Im} \int u \ln |u|^2 \bar{u} = \operatorname{Im} \int |u|^2 \ln |u|^2 = 0,$$

$$(f, \bar{u}) = \operatorname{Im} \int f \bar{u} \leq \|u\|_{L^2} \|f\|_{L^2} \leq \frac{\gamma}{2} \|u\|_{L^2}^2 + \frac{1}{2\gamma} \|f\|_{L^2}^2.$$

Thus, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \gamma \|u\|_{L^2}^2 \leq \frac{\gamma}{2} \|u\|_{L^2}^2 + \frac{1}{2\gamma} \|f\|_{L^2}^2,$$

which implies

$$\frac{d}{dt} \|u\|_{L^2}^2 + 2\gamma \|u\|_{L^2}^2 \leq \gamma \|u\|_{L^2}^2 + \frac{1}{\gamma} \|f\|_{L^2}^2,$$

and consequently

$$\frac{d}{dt} \|u\|_{L^2}^2 \leq \frac{1}{\gamma} \|f\|_{L^2}^2 - \gamma \|u\|_{L^2}^2.$$

Applying Gronwall's inequality yields

$$\|u\|_{L^2}^2 \leq e^{-\gamma t} \|u_0\|_{L^2}^2 + \frac{\|f\|_{L^2}^2}{\gamma^2} (1 - e^{-\gamma t}). \quad (2)$$

□

Lemma 2. Let u be a solution to (1). Then, let

$$J(u) = \|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \|u\|_{L^2}^2 + 2 \int f \bar{u},$$

and

$$F(u) = \|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \int f \bar{u}.$$

Then

$$\frac{d}{dt} J(u) + 2\gamma F(u) = 0.$$

Proof. Multiplying equation (1) by $(\bar{u}_t + \gamma \bar{u})$ and taking the real part, and incorporating integral estimations of singular weight functions [31] as well as non-linear functional analysis techniques [32], we obtain:

$$(i(u_t + \gamma u)(\bar{u}_t + \gamma \bar{u})) + (u \ln(|u|^2), (\bar{u}_t + \gamma \bar{u})) + (\Delta u, (\bar{u}_t + \gamma \bar{u})) = (f, (\bar{u}_t + \gamma \bar{u})).$$

We then observe that

$$(i(u_t + \gamma u)(\bar{u}_t + \gamma \bar{u})) = \operatorname{Re} \int i(u_t + \gamma u)(\bar{u}_t + \gamma \bar{u}) = 0,$$

$$\begin{aligned} (u \ln(|u|^2), (\bar{u}_t + \gamma \bar{u})) &= \operatorname{Re} \int u \bar{u}_t \ln(|u|^2) + \gamma \operatorname{Re} \int u \bar{u} \ln(|u|^2) \\ &= \int \left(\frac{1}{2} \frac{d}{dt} |u|^2 \right) (\ln(|u|^2)) + \gamma \int |u|^2 \ln(|u|^2) \end{aligned}$$

$$\begin{aligned}
 (u \ln(|u|^2), (\bar{u}_t + \gamma \bar{u})) &= \frac{1}{2} \frac{d}{dt} \int |u|^2 \ln(|u|^2) - \frac{1}{2} \int |u|^2 \frac{d}{dt} [\ln(|u|^2)] + \gamma \int |u|^2 \ln(|u|^2) \\
 &= \frac{1}{2} \frac{d}{dt} \int |u|^2 \ln(|u|^2) - \frac{1}{2} \frac{d}{dt} \int |u|^2 + \gamma \int |u|^2 \ln(|u|^2),
 \end{aligned}$$

$$\begin{aligned}
 (\Delta u, (\bar{u}_t + \gamma \bar{u})) &= \operatorname{Re} \int \Delta u \bar{u}_t + \operatorname{Re} \int \gamma \Delta u \bar{u} \\
 &= -\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 - \gamma \int |\nabla u|^2 \\
 &= -\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 - \gamma \|\nabla u\|_{L^2}^2,
 \end{aligned}$$

$$\begin{aligned}
 (f, (\bar{u}_t + \gamma \bar{u})) &= \operatorname{Re} \int f \bar{u}_t + \gamma \operatorname{Re} \int f \bar{u} \\
 &= \frac{d}{dt} \int f \bar{u} + \gamma \int f \bar{u}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 -\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 - \gamma \|\nabla u\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \int |u|^2 \ln(|u|^2) - \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 \\
 + \gamma \int |u|^2 \ln(|u|^2) = \frac{d}{dt} \int f \bar{u} + \gamma \int f \bar{u},
 \end{aligned}$$

Which leads to

$$\begin{aligned}
 \frac{d}{dt} \left(-\frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int |u|^2 \ln(|u|^2) - \frac{1}{2} \|u\|_{L^2}^2 - \int f \bar{u} \right) \\
 + \gamma \left(-\|\nabla u\|_{L^2}^2 + \int |u|^2 \ln(|u|^2) - \int f \bar{u} \right) = 0,
 \end{aligned}$$

and subsequently,

$$\begin{aligned}
 \frac{d}{dt} \left(\|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \|u\|_{L^2}^2 + 2 \int f \bar{u} \right) + 2\gamma \left(\|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \int f \bar{u} \right) \\
 = 0.
 \end{aligned}$$

By setting

$$J(u) = \|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \|u\|_{L^2}^2 + 2 \int f \bar{u},$$

and

$$F(u) = \|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \int f \bar{u},$$

We finally deduce that

$$\frac{d}{dt}J(u) + 2\gamma F(u) = 0.$$

□

Lemma 3. Let u be a solution to equation (1). Then, we have

$$\|\nabla u\|_{L^2}^2 \leq C_1(f, \gamma).$$

Proof. We establish a relation for $J(u)$ and $F(u)$ as follows

$$\begin{aligned} J(u) - F(u) &= \|\nabla u\|_{L^2}^2 - \int |u|^2 \ln(|u|^2) + \|u\|_{L^2}^2 \\ &\quad + 2 \int f \bar{u} - \|\nabla u\|_{L^2}^2 + \int |u|^2 \ln(|u|^2) - \int f \bar{u} \\ J(u) - F(u) &= \|u\|_{L^2}^2 + \int f \bar{u} \\ &\leq \|u\|_{L^2}^2 + \|f\|_{L^2} \|u\|_{L^2} \\ &\leq \|u\|_{L^2} (\|u\|_{L^2} + \|f\|_{L^2}) \\ &\leq \frac{1}{2\gamma} \|u\|_{L^2}^2 + \frac{\gamma}{2} (\|u\|_{L^2} + \|f\|_{L^2})^2. \end{aligned}$$

Multiplying by 2γ , we obtain

$$\begin{aligned} 2\gamma J(u) - 2\gamma F(u) &\leq \|u\|_{L^2}^2 + \gamma^2 (\|u\|_{L^2} + \|f\|_{L^2})^2 \\ &\leq C_1(f, \gamma). \end{aligned}$$

This implies

$$-2\gamma F(u) \leq -2\gamma J(u) + C_1(f, \gamma).$$

Hence, we arrive at

$$\frac{d}{dt}J(u) + 2\gamma J(u) \leq C_1(f, \gamma).$$

Applying Gronwall's Lemma gives

$$J(u) \leq J(u_0)e^{-2\gamma t} + \frac{C_1(f, \gamma)}{2\gamma} (1 - e^{-2\gamma t}).$$

Remark.

$$\begin{aligned} J(u_0) &= \|\nabla u_0\|_{L^2}^2 - \int |u_0|^2 \ln(|u_0|^2) + \|u_0\|_{L^2}^2 + 2 \int f \bar{u}_0 \\ &\leq \|\nabla u_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 + 2\|f\|_{L^2} \|u_0\|_{L^2}. \end{aligned}$$

This proves that $J(u)$ remains bounded. Moreover

$$\|\nabla u\|_{L^2}^2 \leq J(u_0)e^{-2\gamma t} + C(f, \gamma). \quad (3)$$

From estimates (2) and (3), we conclude that the solution is global [4, 22, 35].

□

2.2 Existence of a Bounded Absorbing Set $\mathcal{B}_0$

Proposition 1. There exists a bounded subset $\mathcal{B}_0$ satisfying

$$S(t)\mathcal{B}_0 \subset \mathcal{B}_0, \quad \forall t \geq 0,$$

and such that, for any bounded set $\mathcal{B}$ in $H^1(\mathbb{R})$, there exists a time $t_0(\mathcal{B}) > 0$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall t \geq t_0(\mathcal{B}).$$

Proof. By virtue of Lemmas 1, 2, 3, and following standard techniques for global attractors [27, 1], we find

$$\|u\|_{L^2}^2 \leq e^{-\gamma t} \|u_0\|_{L^2}^2 + \frac{\|f\|_{L^2}^2}{\gamma^2} (1 - e^{-\gamma t}) \quad \forall t \geq 0,$$

and

$$\|\nabla u\|_{L^2}^2 \leq e^{-2\gamma t} J(u_0) + C(f, \gamma) \quad \forall t \geq 0.$$

Consequently, it follows that

$$\|u\|_{H^1}^2 \leq e^{-2\gamma t} J(u_0) + M_1(f, \gamma) \quad \forall t \geq 0,$$

where

$$t_0 = \frac{1}{2\gamma} \ln \left(\frac{J(u_0)}{M_1(f, \gamma)} \right).$$

Hence, there exists a bounded absorbing set $\mathcal{B}_0$, which corresponds to a ball of radius

$$M_2 = e^{-2\gamma t} J(u_0) + M_1(f, \gamma) \quad \text{in } H^1(\mathbb{R}).$$

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