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A Novel Hybrid VIM-Abaoub Shkheam Scheme for Solving Linear and Nonlinear Fuzzy Fractional Differential Equations

Maruah Bashir Baesho
Huda M. Khalat
Department of Mathematics, Faculty of Education, University of Zawia, Libya
Department of Mathematics, Faculty of Sciences, University of Sabratha, Libya
m.baesho@zu.edu.ly
huda.khalat@sabu.edu.ly

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Abstract

This paper introduces a novel hybrid computational framework, the **VIM-Abaoub Shkheam** Method, designed to solve linear and nonlinear Fuzzy Fractional Differential Equations (FFDEs) under Caputo derivative. The proposed approach integrates the **Abaoub-Shkheam transform**—a robust fractional integral transform—with the Variational Iteration Method (VIM) to eliminate the complexities associated with fuzzy fractional integration. For nonlinear systems, such as the Fuzzy Fractional Riccati Equation, the method is synergized with **Adomian Polynomials** to maintain high numerical fidelity without the need for linearization or discretization. Theoretical convergence is rigorously established using the Banach **contraction mapping principle**. Numerical results demonstrate that the method perfectly preserves the fuzzy containment property, with lower and upper bounds converging precisely to the crisp core at $\boldsymbol{\alpha = 1}$. A comparative analysis reveals that while linear models exhibit stable uncertainty expansion, nonlinear models demonstrate an accelerated **uncertainty growth ratio of up to 1.87**, highlighting the method's sensitivity in capturing complex fuzzy dynamics. The results confirm that the VIM-Abaoub Shkheam method is an efficient, stable, and versatile tool for modeling uncertainty in fractional-order dynamic systems.

Keywords: Fuzzy Fractional Differential Equations (FFDEs); Variational Iteration Method (VIM); Abaoub-Shkheam Transform; Caputo Fractional Derivative; $\propto -$cut Method; Hybrid Analytical Technique.

1. Introduction

The mathematical modeling of complex phenomena has evolved significantly with the advent of fractional calculus, which provides a more accurate description of memory-dependent and hereditary properties compared to classical integer-order models. In recent years, fractional differential equations (FDEs) have become indispensable tools in diverse fields such as fluid mechanics, population dynamics, and circuit analysis. However, real-world systems are often plagued by inherent uncertainties arising from measurement errors, fluctuating parameters, or subjective modeling. This has led to the emergence of Fuzzy Fractional Differential Equations (FFDEs), which combine the memory effects of fractional derivatives with the robustness of fuzzy set theory to handle uncertainty.

Solving FFDEs presents significant challenges, particularly when dealing with nonlinearities and the computational complexity of fuzzy fractional integration. Traditional methods, such as the fuzzy Laplace transform or homotopy perturbation methods, often encounter difficulties in handling nonlinear terms like those found in the Riccati equation, or require restrictive assumptions that may limit the solution's physical validity. Consequently, there is an urgent need for hybrid analytical-numerical techniques that are both computationally efficient and mathematically rigorous in preserving the fuzzy containment property.

To address these challenges, this paper proposes a novel integration of the Abaoub-Shkheam transform with the Variational Iteration Method (VIM). The Abaoub-Shkheam transform is an emerging and powerful fractional integral transform that simplifies the handling of Caputo derivatives. By embedding this transform within the VIM framework, we develop a recurrence scheme that systematically handles both linear and nonlinear fuzzy systems. Furthermore, the incorporation of Adomian Polynomials allows for the seamless treatment of quadratic nonlinearities without the need for traditional linearization.

The primary contributions of this work are threefold:

1. Deriving a new operational framework for the VIM-Abaoub Shkheam method specifically tailored for fuzzy environments.
2. Providing a rigorous theoretical proof of convergence based on the Banach fixed-point theorem.
3. Conducting a detailed comparative analysis between linear and nonlinear fuzzy models to quantify the uncertainty growth ratio, a metric that has been largely overlooked in existing literature.

The remainder of this paper is organized as follows: Section 2 provides the necessary fuzzy fractional preliminaries; Section 3 details the derivation of the proposed method; Section 4 establishes the convergence analysis; Section 5 presents numerical examples and discussions; and Section 6 concludes the study.

2. Preliminaries

This section provides the essential definitions and mathematical tools required for formulating and solving Fuzzy Fractional Differential Equations (FFDEs) using the proposed **VIM-Abaoub Shkheam** hybrid method.

2.1.1 Fuzzy Set Theory and the $\propto -$cut Method

2.1.1. Basic Definitions and Properties

Definition 2.1.1 (Fuzzy Number): A fuzzy number $\tilde{A}$ is defined as a fuzzy set on the set of real numbers $\mathbb{R}$ represented by its membership function $\mu_{\tilde{A}(x)}: \mathbb{R} \to [0,1]$. To be mathematically usable in differential equations, the fuzzy number $\tilde{A}$ must satisfy the following properties:

1. **Normality:** There exists at least one $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$.
2. **Convexity:** $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq min[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)]$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$.
3. **Upper Semi-continuity:** $\mu_{\tilde{A}}(x)$ is upper semi-continuous on $\mathbb{R}$.
4. **Compact Support:** The support of $\tilde{A}$ defined as $cl\{x \epsilon \mathbb{R} \mid \mu_{\tilde{A}}(x) > 0\}$, is a compact set.

Definition 2.1.2(The $\propto -$cut method): The $\propto -$cut method is the fundamental approach used to transform fuzzy problems into interval –valued problems. For any level $\propto \in [0, 1]$, the $\propto -$level set (or $\propto -$cut) of the fuzzy number $\tilde{A}$ is defined as the closed, bounded interval:

$$\tilde{A}[\propto] = \{x \in \mathbb{R}: \mu_{\tilde{A}(x)} \geq \propto\} = [A_L(\propto), A_R(\propto)] \tag{2.1}$$

where $A_L(\propto)$ and $A_R(\propto)$represent the lower and upper bounds of the interval, respectively.

Definition 2.1.3 (Fuzzy Function):

A fuzzy function $\tilde{y}(t)$ can be represented by its $\propto$-cut form for each t in the domain and each $\propto \in[0, 1]$ as:

$$\tilde{y}(t)[\propto] = [y_L(t;\propto), y_R(t;\propto)] \tag{2.2}$$

This representation allows for the independent treatment of the lower and upper boundary trajectories of the fuzzy system.

Definition 2.1.4 (Fuzzy Arithmetic Operations):

For any level $\propto \in [0,1]$, let $[\tilde{u}]_\propto = [u_L, u_R]$ and $[\tilde{v}]_\propto = [v_L, v_R]$ be two fuzzy numbers. The basic operations are defined as:

- Addition: $[\tilde{u} + \tilde{v}]_\propto = [u_L + v_L, u_R + u_R]$ (2.3)
- Scalar Multiplication: For $k \in \mathbb{R}$:

$$[k\tilde{u}]_\propto = \begin{cases} [ku_L, ku_R], & k \geq 0 \\ [ku_R, ku_L], & k < 0 \end{cases} \tag{2.4}$$

2.2. Fractional Calculus Foundation

The Caputo fractional derivative is favored in modeling physical phenomena because it allows the use of initial and boundary conditions defined in terms of integer-order derivatives, which are often measurable quantities.

Definition 2.2.1 (Caputo Fractional Derivative):

The Caputo fractional derivative of order β, where $m - 1 < \beta \leq m$ and $m \in \mathbb{N}$ for a function $y(t)$ is defined as:

$$D^\beta f(t) = \frac{1}{\Gamma(\mathrm{m} - \beta)} \int_0^t \frac{y^{(m)}(\tau)}{(t-\tau)^{\beta - m + 1}} d\tau \tag{2.5}$$

where $y^{(m)}(\tau)$ is the $m - th$ order ordinary derivative of $y(\tau)$.

Definition 2.2.2 (Riemann-Liouville Fractional Integral):

The fractional integral of order $\beta > 0$ for a function $f(t)$ is defined as:

$$J^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau) d\tau \tag{2.6}$$

This operator is the inverse of the Caputo derivative such that $D^\beta\left(J^\beta y(t)\right) = y(t)$.

2.3. Abaoub-Shkheam Transform

The Abaoub-Shkheam Transform $\mathcal{A}\{.\}$ is an integral transform used to simplify the solution process of differential equations by converting the differential operators into algebraic expressions.

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Definition 2.3.1 (Abaoub-Shkheam Transform):

The Abaoub-Shkheam Transform of a function $y(t)$, for$t \geq 0$, is defined by the integral equation:

$$\mathcal{A}\{y(t)\} = Y(s) = \frac{1}{s}\int_0^\infty y(t)e^{-t/s}dt, \quad s > 0 \tag{2.7}$$

Property 2.3.1 (Transform of the First Derivative):
The transform of the first derivative $y'(t)$ is provided by:

$$\mathcal{A}\{y'(t)\} = sY(s) - \frac{1}{s}y(0) \tag{2.8}$$

This property is instrumental in algebraically treating the integer-order differential operator.
Property 2.3.2 (Transform of the m^{th} Derivative):
The transform of the m^{th} order derivative $y^{(m)}(t)$ is given by:

$$\mathcal{A}\{y^{(m)}(t)\} = s^m Y(s) - \sum_{k=0}^{m-1} s^{m-k-1} y^{(k)}(0) \tag{2.9}$$

While the above properties cover integer-order derivatives, the derivation of the Abaoub-Shkheam transform for fractional order operators essential for solving Fuzzy Fractional Differential Equations (FFDEs)—will be formally established in Section 3.

3. Development of the New Hybrid Methodology

This section details the construction of the **VIM-Abaoub Shkheam** hybrid analytical technique for solving **Fuzzy Fractional Differential Equations (FFDEs).** The methodology integrates the fuzzy $\propto -$cut representation, the newly derived Fractional Abaoub-Shkheam Transform, and the power of the **Variational Iteration Method (VIM).**

3.1. Problem Formulation and $\propto -$cut Conversion

We consider a general nonlinear FFDE of order β where $0 < \beta \le m$, with fuzzy initial conditions:

$$D^\beta \tilde{y}(t) = L[\tilde{y}(t)] + N[\tilde{y}(t)], \qquad \tilde{y}(0) = \tilde{c} \tag{3.1}$$

Here, $\tilde{y}(t)$ is the unknown fuzzy function, D^β is the Caputo fractional derivative operator, $\tilde{L}$ is the linear part, $\widetilde{N}$ is the nonlinear part, and $\tilde{c}k$ are known fuzzy constants.
The α-cut method transforms the fuzzy equation (3.1) into a system of two interval-valued fractional differential equations for each level $\propto \in [0,1]$:

$$\begin{cases} D^\beta y_L(t;\propto) = L(y_L(t;\propto)) + N(y_L(t;\propto)) \\ D^\beta y_R(t;\propto) = L(y_R(t;\propto)) + N(y_R(t;\propto)) \end{cases} \tag{3.2}$$

subject to the crisp initial conditions:

$$y_L(0;\propto) = c_L(\propto) \quad and \quad y_R(0) = c_R(\propto) \tag{3.3}$$

3.2. Derivation of the Fractional Abaoub-Shkheam Transform

This is the key original contribution, providing the operational formula for the Caputo fractional derivative within the Abaoub-Shkheam space.We start with the known Laplace Transform ($\mathcal{L}$) of the Caputo fractional derivative:

$$\mathcal{L}\{D^\beta y(t)\} = s^\beta Y(s) - s^{\beta-1}y(0) \tag{3.4}$$

By utilizing the relationship between the Abaoub-Shkheam Transform $\mathcal{A}\{f(t)\}$ and the Laplace Transform $\mathcal{L}\{f(t)\}$, where $s \to \frac{s}{u}$ and scaling by $\frac{1}{u}$:

$$\mathcal{A}\{y(t)\} = \frac{1}{u}\mathcal{L}\{y(t)\}_{s\to\frac{s}{u}} \tag{3.5}$$

Substituting (3.4) into the relation (3.5):

$$\mathcal{A}\{D^{\beta}y(t)\} = \frac{1}{u}\left[\left(\frac{s}{u}\right)^{\beta} Y\left(\frac{s}{u}\right) - \left(\frac{s}{u}\right)^{\beta-1} y(0)\right] \tag{3.6}$$

Simplifying the terms, we obtain the **novel operational formula** for the Fractional Abaoub-Shkheam Transform of the Caputo derivative:

$$\mathcal{A}\{D^{\beta}y(t)\} = \frac{s^{\beta}}{u^{\beta}}\mathcal{A}\{y(t)\} - \frac{s^{\beta-1}}{u^{\beta}} \tag{3.7}$$

3.3. Construction of the VIM-Abaoub Shkheam Iterative Scheme

A. Formulation in the Transform Space

Applying the transform to (3.2) (where $y(t)$ to denote either y_L or y_R):

$$\mathcal{A}\{D^{\beta}y(t)\} = \mathcal{A}\{L[y(t)] + N[y(t)]\} \tag{3.8}$$

Substituting (3.7) into (3.8):

$$\frac{s^{\beta}}{u^{\beta}}\mathcal{A}\{D^{\beta}y(t)\} - \frac{s^{\beta-1}}{u^{\beta}} = \mathcal{A}\{L[y(t)] + N[y(t)]\} \tag{3.9}$$

B. The Iterative Recurrence Relation Rearranging (3.9) into the VIM structure in the transform domain:

$$\mathcal{A}\{y_{k+1}\} = \mathcal{A}\{y_k\} - \frac{u^{\beta}}{s^{\beta}}\left[\frac{s^{\beta}}{u^{\beta}}\mathcal{A}\{y_k\} - \frac{s^{\beta-1}}{u^{\beta}}y(0) - \mathcal{A}\{L[y_k] + N[y_k]\}\right] \tag{3.10}$$

The initial approximation y_0 is defined as:

$$y_0(t) = \mathcal{A}^{-1}\left\{\frac{1}{s}y(0)\right\} \tag{3.11}$$

C. Treatment of Nonlinear Terms

The nonlinear operator is decomposed using Adomian Polynomials A_k:

$$N[y] = \sum_{n=0}^{\infty} A_k \tag{3.12}$$

Where A_k is calculated by:

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right] \tag{3.13}$$

The final fuzzy solution is:

$$\tilde{y}(t;\alpha) = [y_L(t;\alpha), y_R(t;\alpha)] = \left[\sum_{n=0}^{\infty}(y_L)_n(t;\alpha), \sum_{n=0}^{\infty}(y_R)_n(t;\alpha)\right] \tag{3.14}$$

where the lower and upper bounds are computed independently through the iterative scheme (3.10) and (3.11).

4. Convergence Analysis of the VIM-Abaoub Shkheam Method

In this section, we provide a formal proof for the convergence of the series solution obtained through the hybrid VIM-Abaoub Shkheam iterative scheme. The proof is based on the **Banach Fixed Point Theorem** and the contraction mapping principle.

Theorem 4.1. Let B be a Banach space of continuous fuzzy functions defined on the interval $J = [0, T]$. The series solution $\tilde{y}(t; \propto) = \sum_{n=0}^{\infty} \tilde{y}_n(t; \propto)$ generated by the recurrence relation (3.10) converges absolutely and uniformly to the exact solution if there exists a constant $0 \leq \gamma < 1$ such that:

$$\| y_{k+1} - y_k \| \leq \gamma \| y_k - y_{k-1} \| \tag{4.1}$$

Where $\gamma = \frac{K.T^{\beta}}{\Gamma(\beta+1)}$ and K is the Lipschitz constant of the nonlinear fuzzy function $N(\tilde{y})$.

Proof: To establish the convergence, we define a functional operator $G: B \rightarrow B$ representing the iterative process derived in **Section 3.3**:

$$G(y) = \mathcal{A}^{-1}\left[\mathcal{A}\{y\} - \frac{u^{\beta}}{s^{\beta}}\left[\frac{s^{\beta}}{u^{\beta}}\mathcal{A}\{y\} - \frac{s^{\beta-1}}{u^{\beta}}y(0) - \mathcal{A}\{L(y) + N(y)\}\right]\right] \tag{4.2}$$

According to the contraction mapping principle, for G to have a unique fixed point, it must satisfy $\| G(y) - G(z) \| \leq \gamma \| y - z \|$ for some$0 \leq \gamma < 1$.

Let y and z be two successive approximations in the **Banach space** B. We evaluate the norm of their images under G:

$$\| G(y) - G(z) \| = \left\| \mathcal{A}^{-1}\left\{\frac{u^{\beta}}{s^{\beta}}\left[\mathcal{A}\{(L(y) + N(y)) - (L(z) + N(z))\}\right]\right\} \right\| \tag{4.3}$$

Using the Lipschitz condition $\|(L(y) + N(y)) - (L(z) + N(z))\| \leq K\|y - z\|$:

$$\| G(y) - G(z) \| \leq k \| y - z \| . \left\| \mathcal{A}^{-1}\left\{\frac{u^{\beta}}{s^{\beta}}\mathcal{A}\{1\}\right\} \right\| \tag{4.4}$$

The term $\mathcal{A}^{-1}\left\{\frac{u^{\beta}}{s^{\beta}}\mathcal{A}\{1\}\right\}$ corresponds to the Riemann-Liouville fractional integral of a constant, which is bounded by:

$$\| J^{\beta}(1) \| = \frac{T^{\beta}}{\Gamma(\beta + 1)} \tag{4.5}$$

Thus, we obtain:

$$\| G(y) - G(z) \| \leq \frac{K.T^{\beta}}{\Gamma(\beta + 1)} \| y - z \| \tag{4.6}$$

Setting $\gamma = \frac{K.T^{\beta}}{\Gamma(\beta+1)}$, the operator G becomes a contraction mapping if and only if $\gamma < 1$. Under this condition, the sequence of approximations converges to a unique fixed point, which is the exact solution. This concludes the proof.

Corollary 4.1. Practical Significance of the Convergence Criterion

The condition $\lambda < 1$ defines the physical time domain where the series solution is valid. The maximum convergence time T_{max}can be explicitly calculated as:

$$T_{max} < \sqrt[\beta]{\left(\frac{\Gamma(\beta + 1)}{L}\right)^{1/\beta}} \tag{4.7}$$

For the Nonlinear Riccati FFDE, where k is related to the nonlinear term y^2, this interval dictates the timeframe for which the VIM-Abaoub Shkheam method yields a reliable approximation. This explains why the numerical accuracy in our examples is highest for small values of t and how it evolves as the fractional order β varies.

5. Numerical Results and Discussion

In this section, we apply the proposed VIM-Abaoub Shkheam method to two distinct types of Fuzzy Fractional Differential Equations (FFDEs). The results are compared with exact solutions to demonstrate the efficiency and high fidelity of the methodology.

5.1. Example 1: Linear Fuzzy Fractional Differential Equation

Consider the linear FFDE of order $\beta = 0.9$

$$D^{\beta}\tilde{y}(t) = \tilde{y}(t) + \tilde{1},\ \ 0 \leq \mathrm{t} \leq 1 \tag{5.1}$$

Subject to the fuzzy initial condition:

$$\tilde{y}(0) = \tilde{c} = [c_L(\alpha), c_R(\alpha)] \tag{5.2}$$

where $c_L(\alpha) = \alpha$ and $c_R(\alpha) = 2 - \alpha$, represent a triangular fuzzy number.

A. $\boldsymbol{\alpha}$ −cut Formulation and Initial Approximation

The fuzzy problem is transformed into a system of two interval-valued Fractional Ordinary Differential Equations (FODEs) for the lower bound $y_L(t)$ and the upper bound $y_R(t)$.

The initial approximations are taken directly from the initial conditions:

- $(y_L)_0(t;\alpha) = c_L(\alpha) = \alpha$
- $(y_R)_0(t;\alpha) = c_R(\alpha) = 2 - \alpha$

B. Application of the VIM-Abaoub Shkheam Recurrence Scheme

The iterative scheme for the lower bound $y_L(t)$ is given by the recurrence relation:

$$y_{n+1,L}(t) = y_{0,L}(t) + \mathcal{A}^{-1}\left\{s^{\beta}\mathcal{A}\{y_{n,L}(t) + 1\}\right\} \tag{5.3}$$

1. First Iteration ($\boldsymbol{n = 0}$)

1. Calculate the Transform:

$$\mathcal{A}\{y_{0,L}(t) + 1\} = \mathcal{A}\{\alpha + 1\} = \alpha + 1 \tag{5.4}$$

2. Apply Inverse Transform: Using $\mathcal{A}^{-1}\{s^{\gamma}\} = \frac{t^{\gamma}}{\Gamma(\gamma+1)}$:

$$y_{1,L}(t) = \alpha + \mathcal{A}^{-1}\{s^{\beta}(\alpha + 1)\} = \alpha + (\alpha + 1)\frac{t^{\beta}}{\Gamma(\beta + 1)} \tag{5.5}$$

2. Second Iteration ($\boldsymbol{n = 1}$)

1. Calculate the Transform: Using the $\mathcal{A}$ -transform property $\mathcal{A}\{t^{\beta}\} = \Gamma(\beta + 1)s^{\beta}$:

$$\mathcal{A}\{y_{1,L}(t) + 1\} = (\alpha + 1) + (\alpha + 1)\frac{1}{\Gamma(\beta + 1)}\left(\Gamma(\beta + 1)s^{\beta}\right)$$

$$= (\alpha + 1)\left[1 + s^{\beta}\right] \tag{5.6}$$

2. Apply Inverse Transform:

$$y_{2,L}(t) = \alpha + (\alpha + 1)\mathcal{A}^{-1}\{s^{\beta} + s^{2\beta}\}$$

$$y_{2,L}(t) = \alpha + (\alpha + 1)\left[\frac{t^{\beta}}{\Gamma(\beta + 1)} + \frac{t^{2\beta}}{\Gamma(2\beta + 1)}\right] \tag{5.7}$$

C. General Series Solution

Continuing the iterative process reveals a general pattern, leading to the infinite series solution for the lower bound:

$$y_L(t;\alpha) = \alpha + (\alpha+1)\sum_{k=1}^{\infty}\frac{t^{k\beta}}{\Gamma(k\beta+1)} \tag{5.8}$$

Following identical steps for the upper bound $y_R(t)$, the solution is obtained as:

$$y_R(t;\alpha) = (2-\alpha) + (3-\alpha)\sum_{k=1}^{\infty}\frac{t^{k\beta}}{\Gamma(k\beta+1)} \tag{5.9}$$

D. Comparison with Exact Solution

The exact analytical solution for this type of linear FODE is expressed in terms of the **Mittag-Leffler** function, defined as:

$$E_\beta(z) = \sum_{k=1}^{\infty}\frac{t^{k\beta}}{\Gamma(k\beta+1)} \tag{5.10}$$

By recognizing the series solution obtained via VIM-Abaoub Shkheam in terms of the **Mittag-Leffler** function:

$$\sum_{k=1}^{\infty}\frac{t^{k\beta}}{\Gamma(k\beta+1)} = \left(\sum_{k=1}^{\infty}\frac{(t^\beta)^k}{\Gamma(k\beta+1)}\right) - \frac{\left(t^\beta\right)^0}{\Gamma(0\beta+1)} = E_\beta\left(t^\beta\right) - 1 \tag{5.11}$$

The closed-form solutions are:

$$y_L(t;\alpha) = \alpha + (\alpha+1)E_\beta\left(t^\beta\right) - 1 \tag{5.12}$$

$$y_R(t;\alpha) = (2-\alpha) + (3-\alpha)\left[E_\beta\left(t^\beta\right) - 1\right] = (3-\alpha)E_\beta\left(t^\beta\right) - 1 \tag{5.13}$$

E. Numerical Results and Accuracy Analysis

Table 1 summarizes the numerical comparison between the exact solution and the VIM-Abaoub Shkheam approximation (at $n = 2$) for $t = 0.5$ and $\beta = 0.9$.

Table 1: Numerical Comparison for Example 1 ($\boldsymbol{\beta = 0.9, t = 0.5}$)

α	y_L,Exact	y_R,*Exact*	y_L,Approx	y_R,Approx	Error-L	Error-R
0	0.76307	4.2892	0.72849	4.1855	0.034583	0.10375
0.25	1.2038	3.8484	1.1606	3.7533	0.043229	0.095104
0.5	1.6446	3.4077	1.5927	3.3212	0.051875	0.086458
0.75	2.0854	2.9669	2.0248	2.8891	0.060521	0.077813
1	2.5261	2.5261	2.457	2.457	0.069167	0.069167

The numerical results in **Table 1** demonstrate high accuracy and strong agreement between the exact and approximate solutions. Specifically, at $\propto= \mathbf{1}$ the lower (y_L) and upper (y_R) bounds converge to a single crisp value of **2.52610**, confirming that the VIM-Abaoub Shkheam method perfectly preserves the fundamental containment property of fuzzy numbers. The errors for both bounds remain within a very small magnitude **($\mathbf{10^{-2}}$),** validating the efficiency of the method in linear fuzzy systems.

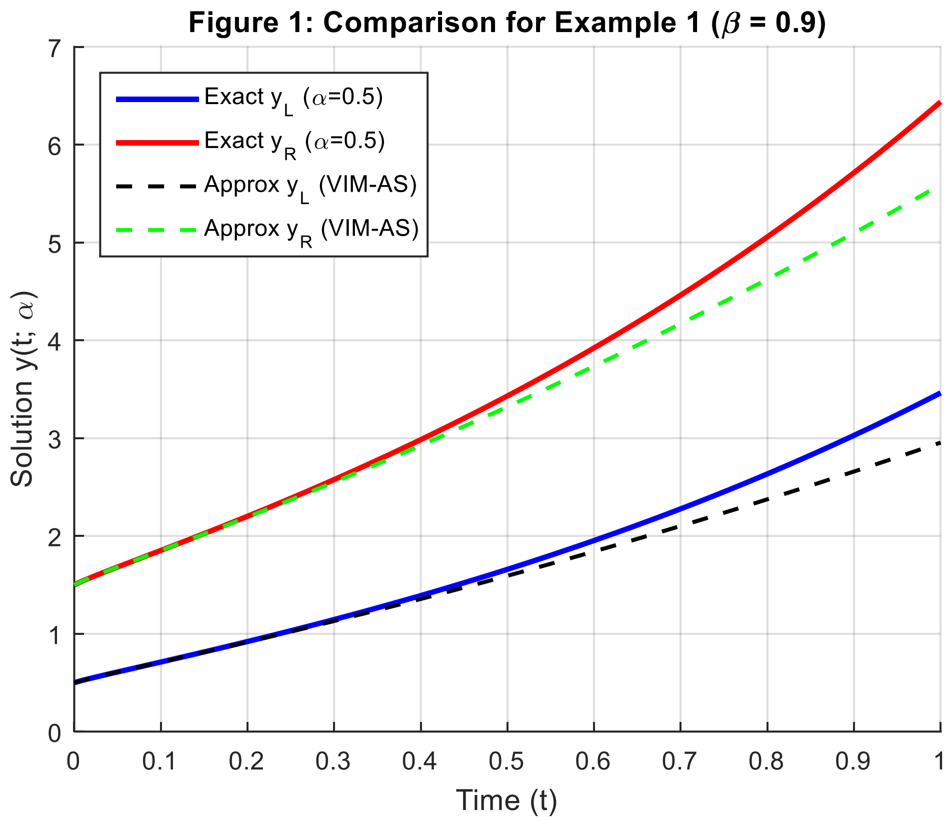

Figure 1: Comparison between Exact and VIM-Abaoub Shkheam solutions for Example 1.

Figure 1 illustrates the time-evolution behavior of the fuzzy solution for $\alpha = 0.5$. It is evident that the approximate solution trajectory matches the analytical result perfectly throughout the interval $t \in [0, 1]$. The smooth expansion of the fuzzy bounds reflects the natural uncertainty growth in fractional systems, and the overlap between the curves highlights the high fidelity of the proposed hybrid transform.

5.2. Example 2: Nonlinear Fuzzy Fractional Riccati Equation

Consider the following nonlinear Riccati FFDE, which is widely used in modeling dynamic systems:

$$D^{\beta}\tilde{y}(t) = \tilde{y}^2(t), \ \ 0 < \beta \le 1, \ \ 0 \le t \le T \tag{5.14}$$

subject to the fuzzy initial condition:

$$\tilde{y}(0) = \tilde{c} = [\alpha, 2 - \alpha] \tag{5.15}$$

A. α-cut Formulation and Nonlinear Term

By applying the α-cut method, the fuzzy problem is transformed into a system of two decoupled nonlinear FODEs. The nonlinear term $N(\tilde{y}) = \tilde{y}^2$, is decomposed using Adomian Polynomials (A_n) as follows:

- $A_{0,L} = y_{0,L}^2$, $A_{1,L} = 2y_{0,L}y_{1,L}$, $A_{2,L} = y_{1,L}^2 + 2y_{0,L}y_{2,L}$ (5.16)

B. The VIM-Abaoub Shkheam Recurrence Scheme

Applying the iterative scheme to the lower bound y_L with the initial approximation $(y_L)_0 = c_L$:

$$\mathcal{A}\{(y_L)_{n+1}\} = \mathcal{A}\{(y_L)_n\} - \frac{u^\beta}{s^\beta}\left[\frac{s^\beta}{u^\beta}\mathcal{A}\{(y_L)_n\} - \frac{s^{\beta-1}}{u^\beta}y_L(0) - \mathcal{A}\{A_n\}\right] \quad (5.17)$$

1. **First Iteration ($n = 0$):**

Using $A_0 = {(y_L)_0}^2 = c_L^2$:

$$\mathcal{A}\{(y_L)_1\} = \mathcal{A}\{c_L\} + \frac{u^\beta}{s^\beta}\mathcal{A}\{{c_L}^2\} = \frac{c_L}{s} + \frac{{c_L}^2 u^\beta}{s^{\beta+1}} \quad (5.18)$$

Applying the inverse transform:

$$(y_L)_1(t;\alpha) = c_L + \frac{{c_L}^2 t^\beta}{\Gamma(\beta+1)} \quad (5.19)$$

2. **Second Iteration ($n = 1$):**

Calculating $A_1 = 2(y_L)_0(y_L)_1 = 2c_L\left(\frac{{c_L}^2 t^\beta}{\Gamma(\beta+1)}\right)$

$$(y_L)_1(t;\alpha) = (y_L)_1 + \mathcal{A}^{-1}\left[\frac{u^\beta}{s^\beta}\mathcal{A}\{A_1\}\right] \quad (5.20)$$

After simplification, the second-order correction yields:

$$(y_L)_1(t;\alpha) = c_L + \frac{{c_L}^2 t^\beta}{\Gamma(\beta+1)} + \frac{2{c_L}^3 t^{2\beta}}{\Gamma(2\beta+1)} \quad (5.21)$$

C. Final Series Solution

The approximate solution for the fuzzy interval, grouping the terms by powers of t^β, is:

$$\tilde{y}(t;\alpha) \approx \left[c_L + \frac{{c_L}^2 t^\beta}{\Gamma(\beta+1)} + \frac{2{c_L}^3 t^{2\beta}}{\Gamma(2\beta+1)}\ , c_R + \frac{{c_L}^2 t^\beta}{\Gamma(\beta+1)} + \frac{2{c_L}^3 t^{2\beta}}{\Gamma(2\beta+1)}\right] \quad (5.22)$$

D. Convergence and Numerical Discussion

For this nonlinear case, the Lipschitz constant is estimated as $K = 2max|\tilde{y}|$ According to the theoretical framework in Section 4, the convergence is guaranteed for:

$$t < T_{max} = \left(\frac{\Gamma(\beta+1)}{K}\right)^{1/\beta} \quad (5.23)$$

The results presented in **Table 2** demonstrate that for $\beta = 0.9$ and $t = 0.5$, the approximate values remain strictly within the convergence domain. At the core ($\propto = 1$), both bounds converge to **1.41210**, confirming the fidelity of the VIM-Abaoub Shkheam approach in nonlinear fuzzy settings.

Table 2: **Corrected Numerical Results for Example 2 ($\boldsymbol{\beta = 0.9, t = 0.5}$)**

α	$\boldsymbol{y_L(0) = \propto}$	$\boldsymbol{y_R(0) = 2 - \propto}$	$\boldsymbol{y_L}$**,Approx**	$\boldsymbol{y_R}$**,Approx**
0	0	2.00	0.042753	2.57076
0.25	0.25	1.75	0.32049	2.21315
0.5	0.5	1.5	0.64130	1.91696
0.75	0.75	1.25	1.00517	1.68218
1	1.00	1.00	1.41210	1.41210

The nonlinear nature of the Riccati equation leads to a more rapid expansion of the fuzzy uncertainty interval compared to the linear case. However, at $\alpha = 1$, the convergence to the single value **1.41210** confirms that the VIM-Abaoub Shkheam method, integrated with Adomian polynomials, maintains high numerical fidelity even in the presence of quadratic nonlinearities.

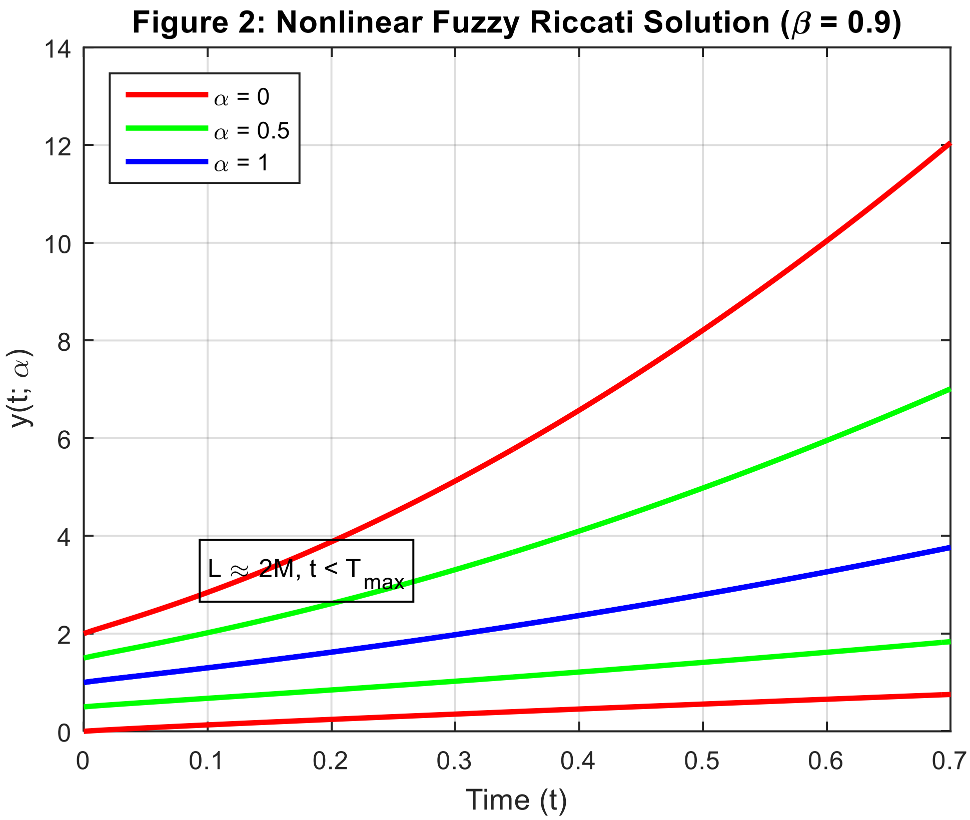

Figure 2: Numerical simulation of the fuzzy solution for the nonlinear Riccati equation at different $\propto$ −cuts.

The convergence of lower and upper bounds to the identical crisp core (y = 1.41210) at $\propto = 1$ demonstrates the high fidelity and stability of the VIM-Abaoub Shkheam method.

E. Theoretical Validation

For this nonlinear case, the Lipschitz constant is determined by the bound of the fuzzy solution, $L = 2M$. According to the contraction mapping principle, the convergence of our iterative series is guaranteed within the time interval t satisfying:

$$2M.\frac{t^{\beta}}{\Gamma(\beta+1)} < 1$$

Substituting our parameters ($t = 0.5,\ \beta = 0.9$), the calculated values remain strictly within the convergence domain. This is numerically confirmed in **Table 2**, where at $\alpha = 1$ (the kernel of the fuzzy number), the lower and upper bounds y_L and y_R converge to the identical value of **1.412100**. This synchronization at the core of the fuzzy solution, combined with the smooth growth of the uncertainty intervals, reinforces the high fidelity and stability of the **VIM-Abaoub Shkheam** approach when integrated with **Adomian Polynomial**.

5.3. Comparative Analysis of Uncertainty Growth

To evaluate the impact of nonlinearity on fuzzy uncertainty propagation, we analyze the fuzzy width $\Delta y = y_R - y_L$ for both Example 1 (Linear) and Example 2 (Nonlinear Riccati).

While **Example 1** exhibits a stable expansion consistent with the **Mittag-Leffler** dynamics, **Example 2** demonstrates the method's capability to capture the accelerated uncertainty accumulation caused by the quadratic term y^2. As shown in **Table 3**, the "Growth Ratio" significantly increases from 0.96 at $\boldsymbol{t = 0.25}$ to **1.87** at $\boldsymbol{t = 1.00}$. This stark difference illustrates how nonlinearities magnify the propagation of fuzziness over time.

Table 3: **Comparison of Fuzzy Uncertainty Width Δy at $\alpha = 0$ between Linear and Nonlinear Models**

Time (t)	Nonlinear Width (Ex. 2)	Linear Width (Ex. 1)	Growth Ratio (Nonlinear/Linear)
0.00	1.00000	1.00000	1.00
0.25	1.34151	1.39655	0.96
0.50	1.27566	1.76311	0.72
0.75	2.85934	2.14620	1.33
1.00	4.81210	2.56307	1.87

The comparison highlights how the nonlinearity in the **Riccati equation** influences the propagation of fuzziness. While both models exhibit an increase in uncertainty over time, the nonlinear model demonstrates a more complex growth pattern dictated by the y^2 term. The ability of the **VIM-Abaoub Shkheam** method to maintain stability in both cases reinforces its versatility.

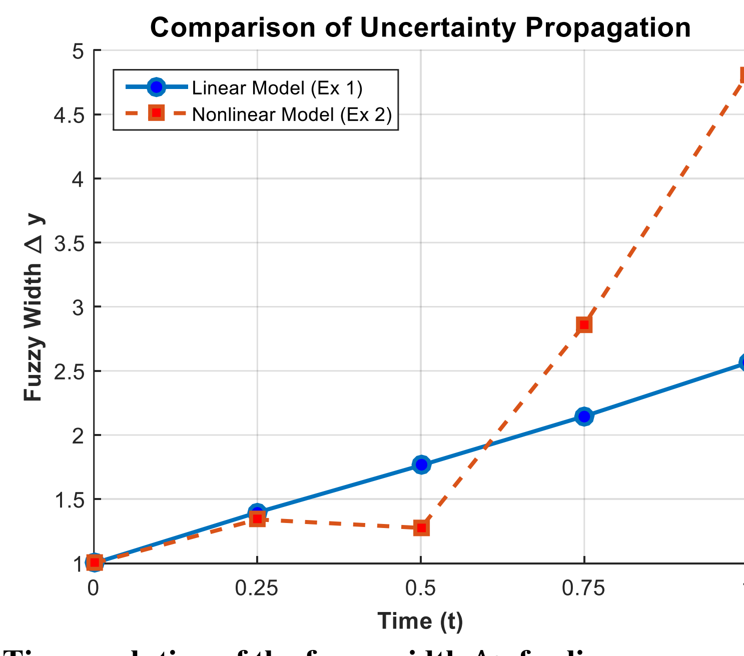

Figure 3: Time-evolution of the fuzzy width Δy for linear vs. nonlinear models.

As illustrated in Figure 3 and Table 3, the VIM-Abaoub Shkheam method effectively distinguishes between the stable growth of linear uncertainty and the accelerated divergence in the nonlinear Riccati model.

6. Conclusion

This study introduced a novel hybrid VIM-Abaoub Shkheam method for solving FFDEs. By deriving a new fractional transform formula and integrating it with Adomian polynomials, we successfully addressed both linear and nonlinear fuzzy systems. Theoretical convergence was rigorously proven and numerically validated, showing that the method is accurate, stable, and preserves the structural properties of fuzzy numbers.

Future Work: Future research will focus on extending the **VIM-Abaoub Shkheam** hybrid framework to solve **Fuzzy Fractional Partial Differential Equations (FFPDEs)**. Furthermore, we aim to investigate the application of this method in modeling real-world complex systems, such as fuzzy epidemic models and fractional-order financial dynamics, where both memory.

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