Comprehensive Journal of Science
Volume (11), Issue (41), (JUNE, 2026)
ISSN: 3014-6266

مجلة العلوم الشاملة
المجلد (11) ملحق العدد 41 يونيو 2026
ردمد: 3014-6266

A Comparative Analytical Study of the Solutions of One-Dimensional Heat and Wave Equations via Separation of Variables and Fourier Series

Mofeda Alajili Ahmed Neser
Department of Mathematics - Faculty of Engineering - University of Zawiya
m.neser@zu.edu.ly
https://orcid.org/0009-0008-9167-0267

Received: 16/05/2026 - Revised: 18/06/2026 - Accepted: 28/06/2026 - Published: 17/07/2026

Abstract

This research provides a comparative analytical examination of the one-dimensional heat and wave equations through the method of separation of variables. Both equations are analyzed on a finite interval with homogeneous Dirichlet boundary conditions. This framework results in an identical spatial eigenvalue problem and a common set of orthogonal sine eigenfunctions. The primary difference between the two models manifests in the temporal component of the separated solutions. In the case of the heat equation, each Fourier mode is accompanied by an exponentially decaying factor, which facilitates diffusion, smoothing, and convergence towards equilibrium. Conversely, the wave equation produces oscillatory sine and cosine temporal factors, leading to sustained modal motion in the absence of damping.

The study derives the analytical Fourier sine series solutions for both equations and contrasts their temporal behavior, long-term behavior, and handling of high-frequency modes. To substantiate the theoretical results, numerical and graphical representations based on truncated Fourier series are provided. The findings indicate that, despite the heat and wave equations possessing the same spatial eigenfunctions under the specified boundary conditions, their qualitative behaviors are inherently different. This distinction is influenced by the order and nature of the time derivative: first order in the heat equation and second order in the wave equation. The comparison elucidates the mathematical differences between parabolic and hyperbolic partial differential equations and underscores the divergence between diffusion and wave propagation.

Keywords: Heat equation; Wave equation; Separation of variables; Fourier sine series; Eigenvalue problem; Orthogonal eigenfunctions; Long-time behavior; High-frequency modes.

1. Background and Research Problem

Partial differential equations serve as a crucial mathematical framework for characterizing physical phenomena that are influenced by both spatial and temporal variables [1]. Notable classical models in this domain include the one-dimensional heat equation and the one-dimensional wave equation, which emerge naturally in the examination of diffusion, heat transfer, vibrations, and wave propagation [12]. Despite the fact that these two equations pertain to distinct physical processes, they can be investigated within a shared mathematical context, facilitating a direct comparison of their solution structures [4].

In this research, both equations are analyzed on a finite interval

$$0 < x < L, t > 0,$$

under homogeneous Dirichlet boundary conditions. The one-dimensional heat equation is given by

$$u_t = \kappa u_{xx}, \kappa > 0,$$

where $u(x,t)$represents the temperature distribution and $\kappa$is the thermal diffusivity. The one-dimensional wave equation is written as

$$v_{tt} = c^2 v_{xx}, c > 0,$$

where $v(x,t)$represents the displacement of a vibrating medium and $c$is the wave speed.

A standard analytical method for solving such linear boundary value problems is the method of separation of variables [3]. This method assumes that the solution can be expressed as a product of a spatial function and a temporal function, for example

$$w(x,t) = X(x)T(t).$$

Under the homogeneous boundary conditions

$$X(0) = 0, X(L) = 0,$$

both the heat equation and the wave equation lead to the same spatial eigenvalue problem

$$X'' + \lambda X = 0, X(0) = X(L) = 0.$$

The corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots.$$

These eigenfunctions form the basis for constructing Fourier sine series solutions for both equations [2].

Despite this common spatial structure, [4] the temporal components of the separated solutions are fundamentally different. For the heat equation, the time-dependent function satisfies

$$T' + \kappa\lambda T = 0,$$

which gives an exponentially decaying factor. For the wave equation, the corresponding temporal equation is

$$T'' + c^2\lambda T = 0,$$

which results in oscillatory sine and cosine components. Consequently, the same category of spatial eigenfunctions yields two different forms of time evolution [1].

The research issue tackled in this investigation is thus to elucidate, through a cohesive separation-of-variables approach, the reason why the one-dimensional heat equation generates dissipative solutions, whereas the one-dimensional wave equation results in oscillatory solutions. More specifically, the research examines how the distinction between the temporal equations influences the disparity between diffusion and wave propagation.

In this context, the primary inquiry of this study is:

How does the temporal aspect of the separated solution clarify the essential difference between exponential decay in the one-dimensional heat equation and enduring oscillation in the one-dimensional wave equation?

2. Objectives and Scope of the Study

The primary aim of this research is to provide a thorough analytical comparison of the one-dimensional heat and wave equations through the application of the separation of variables technique. This study seeks to illustrate how two equations, which share the same spatial eigenvalue problem, can yield distinct temporal behaviors.
More specifically, the goals of the research are:
1. To establish the one-dimensional heat and wave equations on a finite interval while adhering to homogeneous Dirichlet boundary conditions.
2. To derive the relevant Fourier sine series solutions utilizing the eigenvalues and eigenfunctions acquired from the corresponding spatial boundary value problem.
3. To examine and contrast the temporal behavior of the separated solutions, with a particular focus on exponential decay in the heat equation and oscillatory motion in the wave equation.
4. To elucidate the mathematical differences between diffusion and wave propagation as demonstrated by the structure of the resulting analytical solutions.
The study's scope is confined to linear one-dimensional problems defined within the interval.

$$0 < x < L, t > 0.$$

The coefficients $\kappa > 0$and $c > 0$are assumed to be constant. The heat equation is considered with one initial condition, while the wave equation is considered with initial displacement and initial velocity. The boundary conditions are assumed to be homogeneous Dirichlet conditions at both endpoints.
The initial functions are assumed to be sufficiently regular so that they can be represented by Fourier sine series on $(0، L)$. The study focuses on analytical solutions obtained by separation of variables and does not include nonlinear effects, source terms, external forcing, damping, variable coefficients, or nonhomogeneous boundary conditions.

3. Mathematical Preliminaries

This section introduces the mathematical tools required for the analytical treatment of the heat and wave equations. Throughout the study, the spatial domain is the finite interval $(0، L)$, and the time variable satisfies $t > 0$. The functions considered are assumed to have sufficient regularity for the required differentiations, integrations, and Fourier series expansions to be valid.
For functions defined on $(0، L)$, the standard inner product in $L^2(0, L)$is defined by [5]

$$\langle f, g \rangle = \int_0^L f(x)g(x)\,dx,$$

with the associated norm

$$\| f \|_{L^2(0,L)} = \left(\int_0^L | f(x) |^2 \ dx \right)^{1/2}.$$

This inner product is used to express the orthogonality of the sine functions that arise from homogeneous Dirichlet boundary conditions. In particular, the functions [6]

$$\sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots,$$

satisfy the orthogonality relation

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases}$$

Consequently, a suitable function $f(x)$defined on $(0, L)$can be represented by a Fourier sine series of the form [6].

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier sine coefficients are given by [6]:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

For functions in $L^2(0, L)$, this expansion is understood in the L^2-sense, while stronger forms of convergence may be obtained under additional smoothness assumptions on f [5].

The homogeneous Dirichlet boundary conditions used in this study are expressed as

$$w(0,t) = 0, w(L,t) = 0.$$

These conditions naturally lead to sine eigenfunctions in the spatial variable. Thus, Fourier sine series provide the appropriate basis for constructing the separated solutions of both the heat and wave equations [7]. The same orthogonal structure will be used for the two models, while the distinction between them will appear in the corresponding time-dependent equations.

4. Separation of Variables and Eigenvalue Problem

The method of separation of variables is used to reduce the given partial differential equations into ordinary differential equations in the spatial and temporal variables [3]. For a one-dimensional problem, the solution is assumed to have the product form

$$w(x,t) = X(x)T(t),$$

where $X(x)$is the spatial component and $T(t)$is the temporal component. This method is particularly suitable for linear problems with homogeneous boundary conditions [7]. For the heat equation

$$u_t = \kappa u_{xx},$$

we assume

$$u(x,t) = X(x)T(t).$$

Substitution gives

$$X(x)T'(t) = \kappa X''(x)T(t).$$

Dividing by $\kappa X(x)T(t)$, where $X(x)T(t) \neq 0$, yields

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left-hand side depends only on $t$and the right-hand side depends only on x, both sides must be equal to a constant. We choose the separation constant as $-\lambda$. Hence,

$$X'' + \lambda X = 0,$$

and

$$T' + \kappa\lambda T = 0.$$

For the wave equation

$$v_{tt} = c^2 v_{xx},$$

we similarly assume

$$v(x,t) = X(x)T(t).$$

Substitution gives

$$X(x)T''(t) = c^2 X''(x)T(t),$$

and therefore

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Thus, the same spatial equation is obtained,

$$X'' + \lambda X = 0,$$

while the temporal equation becomes

$$T'' + c^2\lambda T = 0.$$

The homogeneous Dirichlet boundary conditions require

$$X(0) = 0, X(L) = 0.$$

Therefore, the spatial eigenvalue problem is [8]

$$X'' + \lambda X = 0, 0 < x < L,$$

with

$$X(0) = 0, X(L) = 0.$$

Nontrivial solutions exist only for positive eigenvalues [8]. Let

$$\lambda = \mu^2, \mu > 0.$$

Then

$$X'' + \mu^2 X = 0,$$

whose general solution is

$$X(x) = A\cos(\mu x) + B\sin(\mu x).$$

Using $X(0) = 0$gives

$$A = 0.$$

Thus,

$$X(x) = B\sin(\mu x).$$

The condition $X(L) = 0$gives

$$B\sin(\mu L) = 0.$$

For a nontrivial solution, $B \neq 0$, and hence

$$\sin(\mu L) = 0.$$

Therefore,

$$\mu L = n\pi, n = 1,2,3, \dots.$$

It follows that the eigenvalues are [8]

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3, \dots,$$

and the corresponding eigenfunctions are [8]

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Thus, both the heat equation and the wave equation share the same spatial eigenfunctions [7]. The essential difference between the two models appears in the temporal equations: the heat equation leads to first-order exponential decay, whereas the wave equation leads to second-order oscillatory motion [1].

5. Analytical Solution of the Heat Equation

Consider the one-dimensional heat equation

$$u_t = \kappa u_{xx}, 0 < x < L, t > 0,$$

subject to the homogeneous Dirichlet boundary conditions

$$u(0,t) = 0, u(L,t) = 0,$$

and the initial condition

$$u(x,0) = f(x), 0 < x < L.$$

Using the separation of variables framework established above, the separated solution is written as

$$u(x,t) = X(x)T(t).$$

For the heat equation, the corresponding spatial and temporal equations are

$$X'' + \lambda X = 0,$$

and

$$T' + \kappa\lambda T = 0.$$

From the homogeneous boundary conditions, the admissible eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots.$$

For each eigenvalue λ_n, the temporal equation becomes

$$T_n' + \kappa\lambda_n T_n = 0.$$

Its solution is

$$T_n(t) = C_n e^{-\kappa\lambda_n t}.$$

Thus, each separated solution has the form

$$u_n(x,t) = C_n e^{-\kappa\lambda_n t}\sin\left(\frac{n\pi x}{L}\right).$$

By the principle of superposition for linear equations, the solution can be expressed as a Fourier sine series [10]:

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\kappa\lambda_n t}\sin\left(\frac{n\pi x}{L}\right).$$

The coefficients $b_n$are determined from the initial condition. Setting $t = 0$gives

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Using the orthogonality of the sine eigenfunctions, the Fourier coefficients are

$$b_n = \frac{2}{L}\int_0^L f(s)\sin\left(\frac{n\pi s}{L}\right) ds.$$

Therefore, the analytical solution of the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(s)\sin\left(\frac{n\pi s}{L}\right) ds\right] e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right).$$

This formula shows that the initial temperature distribution is decomposed into sine modes, and each mode decays exponentially in time at a rate determined by the corresponding eigenvalue [9].

6. Qualitative Behavior of the Heat Equation Solution

The solution of the heat equation can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\kappa\lambda_n t}\sin\left(\frac{n\pi x}{L}\right), \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

This representation shows that the time evolution of each Fourier mode is governed by the exponential factor

$$e^{-\kappa\lambda_n t}.$$

Since $\kappa > 0$and $\lambda_n > 0$, each mode decays as time increases. Moreover, because $\lambda_n$grows like n^2, higher-frequency modes decay faster than lower-frequency modes. Consequently, rapid oscillations in the initial data are strongly damped, and the solution becomes smoother for $t > 0$.[11]

The decay of the solution may also be observed through its $L^2(0,L)$-norm. Using the orthogonality of the sine functions,

$$\| u(\cdot,t) \|^2_{L^2(0,L)} = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2\, e^{-2\kappa\lambda_n t}.$$

This expression is nonincreasing in time and tends to zero under homogeneous Dirichlet boundary conditions [13]. Hence,

$$\lim_{t\to\infty} u(x,t) = 0,$$

for appropriate initial data. The zero limiting state is associated with thermal equilibrium when both ends of the rod are kept at zero temperature [9].

The parameter κ governs the diffusion rate. A higher value of κ results in a quicker decay of all modes, whereas a lower value delays the approach to equilibrium. Consequently, the qualitative characteristics of the heat equation are defined by exponential decay, smoothing of the initial data, attenuation of high-frequency components, and convergence towards equilibrium [11]. These attributes illustrate the dissipative and parabolic characteristics of the heat equation [14].

7. Analytical Solution of the Wave Equation

Consider the one-dimensional wave equation

$$v_{tt} = c^2 v_{xx}, 0 < x < L, t > 0,$$

subject to the homogeneous Dirichlet boundary conditions

$$v(0,t) = 0, v(L,t) = 0,$$

and the initial conditions

$$v(x,0) = f(x), v_t(x,0) = g(x), 0 < x < L.$$

Using the separation of variables framework, assume

$$v(x,t) = X(x)T(t).$$

Substitution into the wave equation gives the spatial and temporal equations

$$X'' + \lambda X = 0,$$

and

$$T'' + c^2 \lambda T = 0.$$

From the homogeneous boundary conditions, the eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots.$$

For each eigenvalue λ_n, the temporal equation becomes

$$T_n'' + c^2 \lambda_n T_n = 0.$$

Thus,

$$T_n'' + \left(\frac{cn\pi}{L}\right)^2 T_n = 0.$$

The general solution is

$$T_n(t) = A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right).$$

Therefore, by the principle of superposition, the solution of the wave equation is represented as the Fourier sine series [12]:

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right).$$

The coefficients $A_n$and $B_n$are determined from the initial displacement and initial velocity. Setting $t = 0$gives

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

Hence,

$$A_n = \frac{2}{L}\int_0^L f(s)\sin\left(\frac{n\pi s}{L}\right) ds.$$

Next, differentiating the solution with respect to t, we obtain

$$v_t(x,t) = \sum_{n=1}^{\infty} \left[-A_n \frac{cn\pi}{L} \sin\left(\frac{cn\pi t}{L}\right) + B_n \frac{cn\pi}{L} \cos\left(\frac{cn\pi t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right).$$

Using the initial velocity condition $v_t(x,0) = g(x)$, we get

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{L} \sin\left(\frac{n\pi x}{L}\right).$$

Thus,

$$B_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(s)\sin\left(\frac{n\pi s}{L}\right) ds,$$

and therefore

$$B_n = \frac{2}{cn\pi}\int_0^L g(s)\sin\left(\frac{n\pi s}{L}\right) ds.$$

Consequently, the analytical solution is

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right),$$

where

$$A_n = \frac{2}{L}\int_0^L f(s)\sin\left(\frac{n\pi s}{L}\right) ds,$$

and

$$B_n = \frac{2}{cn\pi}\int_0^L g(s)\sin\left(\frac{n\pi s}{L}\right) ds.$$

This solution expresses the displacement as a superposition of standing wave modes. Each mode has the spatial shape

$$\sin\left(\frac{n\pi x}{L}\right)$$

and oscillates in time with angular frequency

$$\omega_n = \frac{cn\pi}{L}.$$

8. Qualitative Behavior of the Wave Equation Solution

The analytical solution of the wave equation is given by

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

This representation shows that the solution is a superposition of standing wave modes [12]. Each mode has the spatial form

$$\sin\left(\frac{n\pi x}{L}\right),$$

and evolves in time through the oscillatory factors

$$\cos\left(\frac{cn\pi t}{L}\right) \text{and} \sin\left(\frac{cn\pi t}{L}\right).$$

Thus, unlike the heat equation, the wave equation does not contain an exponential damping factor. In the ideal homogeneous case considered here, the modal amplitudes are not naturally reduced by the equation itself [13]. Instead, each mode continues to oscillate in time.

The angular frequency of the n-th mode is [12]

$$\omega_n = \frac{cn\pi}{L}.$$

Consequently, higher modes exhibit more rapid oscillations due to the linear increase of ωn with respect to n. Nevertheless, the impact of each mode is contingent upon the coefficients An and Bn, which are established by the initial displacement $f(x)$ and the initial velocity $g(x)$, respectively.

A fundamental qualitative characteristic of the homogeneous wave equation is the principle of energy conservation. For solutions that are sufficiently smooth and adhere to the fixed-end boundary conditions, the total energy is defined as

$$E(t) = \frac{1}{2}\int_0^L [v_t^2(x,t) + c^2 v_x^2(x,t)]\, dx.$$

In the absence of damping and external forcing, this energy remains constant in time [13]:

$$E(t) = E(0).$$

This conservation property elucidates the reason why the solution does not diminish to zero in general; rather, it demonstrates a continuous oscillatory motion [13].

Consequently, the solution to the wave equation is defined by modal oscillation, the persistence of energy, reliance on both initial displacement and initial velocity, and the lack of intrinsic damping. These characteristics highlight the conservative and hyperbolic nature of the wave equation [14].

9. Comparative Analysis of the Two Solutions

The analytical solutions obtained for the heat and wave equations indicate that both models possess an identical spatial structure; however, they fundamentally differ in their temporal behavior [19]. When applying homogeneous Dirichlet boundary conditions on (0ⓜ,L), both equations yield the same eigenfunctions.

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots.$$

Thus, the spatial part of the solution is not the source of the main difference between the two models. Instead, the distinction arises from the time-dependent equations associated with each Fourier mode.

For the heat equation, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\kappa\lambda_n t} X_n(x), \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Each mode is multiplied by the exponentially decaying factor

$$e^{-\kappa\lambda_n t}.$$

Since $\kappa > 0$and $\lambda_n > 0$, all modes decay as $t$increases. Moreover, because $\lambda_n$grows quadratically with n, high-frequency modes decay more rapidly than low-frequency modes. This produces smoothing and drives the solution toward equilibrium [11].

For the wave equation, the solution is

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(c\sqrt{\lambda_n} t\right) + B_n \sin\left(c\sqrt{\lambda_n} t\right)\right] X_n(x).$$

In this context, the temporal factors are depicted by sine and cosine functions. These factors demonstrate oscillatory behavior and remain unchanged over time. As a result, in the absence of damping or external forces, the wave equation sustains oscillatory motion rather than permitting it to dissipate [13]. The -th mode oscillates at an angular frequency.

$$\omega_n = c\sqrt{\lambda_n} = \frac{cn\pi}{L}.$$

Consequently, high-frequency modes oscillate at a faster rate; however, they are not inherently damped by the equation itself.

The distinction between the two solutions is also evident in the necessary initial data. The heat equation is first order in time, thus necessitating only a single initial condition,

$$u(x,0) = f(x).$$

In contrast, the wave equation is second order in time and requires two initial conditions,

$$v(x,0) = f(x), v_t(x,0) = g(x).$$

This difference indicates that the heat equation evolves from an initial distribution, while the wave equation evolves from both an initial displacement and an initial velocity [10].

The main comparative features are summarized in Table 1.

Table 1. Comparative features of the heat and wave equations

Feature	**Heat Equation**	**Wave Equation**
Model equation	$u_t = \kappa u_{xx}$	$v_{tt} = c^2 v_{xx}$
Type of PDE	Parabolic	Hyperbolic
Spatial eigenfunctions	$\sin\left(\frac{n\pi x}{L}\right)$	$\sin\left(\frac{n\pi x}{L}\right)$
Temporal factor	$e^{-\kappa\lambda_n t}$	$\cos\left(c\sqrt{\lambda_n} t\right)$, $\sin\left(c\sqrt{\lambda_n} t\right)$
Time behavior	Exponential decay	Periodic oscillation
Initial data	One initial condition	Two initial conditions
High-frequency modes	Rapidly damped	Rapidly oscillatory
Long-time behavior	Approaches equilibrium	Persistent motion in the undamped case

Physical interpretation	Diffusion and smoothing	Wave propagation and vibration

The analysis indicates that the two equations are intricately linked via their shared spatial eigenvalue problem; however, their temporal frameworks yield distinctly different qualitative behaviors. The heat equation illustrates a dissipative process characterized by the gradual loss of energy through diffusion, whereas the wave equation exemplifies a conservative process where energy oscillates between kinetic and potential forms [19]. Consequently, the mathematical differentiation between the two models is primarily dictated by the form and order of the time derivative: first order in the heat equation and second order in the wave equation [14].

10. Numerical and Graphical Illustrations

To illustrate the analytical results, truncated Fourier sine series are used [15]. This approach does not introduce a separate numerical discretization scheme; rather, it evaluates finite-term approximations of the exact series solutions [16]. The purpose of this section is to visualize the difference between the diffusive behavior of the heat equation and the oscillatory behavior of the wave equation.

For simplicity, the following normalized parameters are used:

$$L = 1, \kappa = 1, c = 1.$$

The same initial profile is chosen for both equations:

$$f(x) = x(1 - x), 0 < x < 1.$$

For the wave equation, the initial velocity is assumed to be zero:

$$g(x) = 0.$$

This choice allows the two solutions to be compared under the same initial displacement or temperature distribution.

The Fourier sine coefficients of $f(x) = x(1 - x)$on $(0, 1)$are

$$b_n = 2\int_0^1 x(1 - x)\sin(n\pi x)\,dx.$$

Evaluating the integral gives

$$b_n = \frac{4(1 - (-1)^n)}{n^3\pi^3}, n = 1,2,3, \dots.$$

Hence,

$$b_n = \begin{cases} \dfrac{8}{n^3\pi^3}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Using the first Nterms, the truncated heat equation solution is

$$u_N(x,t) = \sum_{n=1}^{N} b_n\, e^{-n^2\pi^2 t}\sin(n\pi x).$$

For the wave equation, since $g(x) = 0$, the truncated solution becomes

$$v_N(x,t) = \sum_{n=1}^{N} b_n \cos(n\pi t)\sin(n\pi x).$$

In the graphical comparison, the solutions are evaluated for selected times, for example

$$t = 0, t = 0.05, t = 0.10, t = 0.25.$$

A suitable value such as $N = 50$may be used to obtain accurate plots, since the coefficients $b_n$decay as n^{-3} [18]. The resulting graphs should display $u_N(x,t)$and $v_N(x,t)$on the interval $0 < x < 1$for the selected values of t [17].

For the heat equation, the plots show that the initial profile decreases in amplitude as time increases. The solution becomes smoother and gradually approaches the zero equilibrium state. This behavior is caused by the exponential factor

$$e^{-n^2\pi^2 t},$$

which strongly damps the higher Fourier modes.

For the wave equation, the plots show that the solution does not decay. Instead, the initial profile evolves into an oscillatory motion governed by the factor

$$\cos(n\pi t).$$

The modal amplitudes persist even when damping is absent, while the signs and phases of the modes vary periodically over time.

Consequently, the graphical outcomes corroborate the analytical findings. The heat equation demonstrates diffusion, smoothing, and a tendency to decay towards equilibrium, in contrast to the wave equation, which shows continuous oscillation. Despite both equations commencing from an identical initial profile and employing the same sine eigenfunctions, their time-dependent factors yield distinctly different solution behaviors.

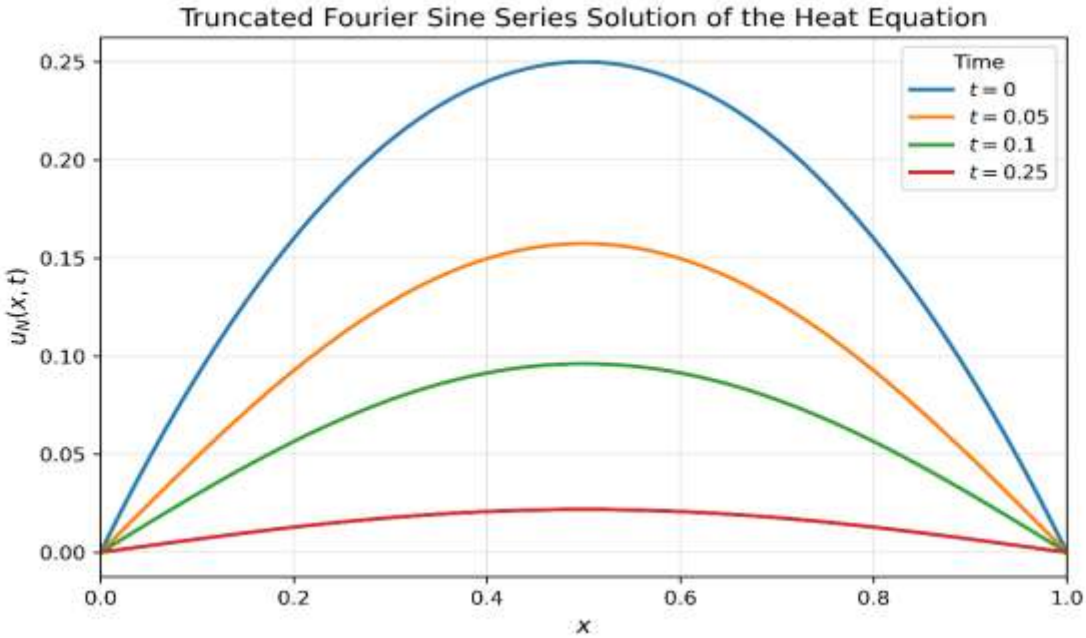

Figure 1. Truncated Fourier sine series solution $u_N(x,t)$of the heat equation for $f(x) = x(1-x)$, $L = 1$, $\kappa = 1$, and $N = 50$, at selected times.

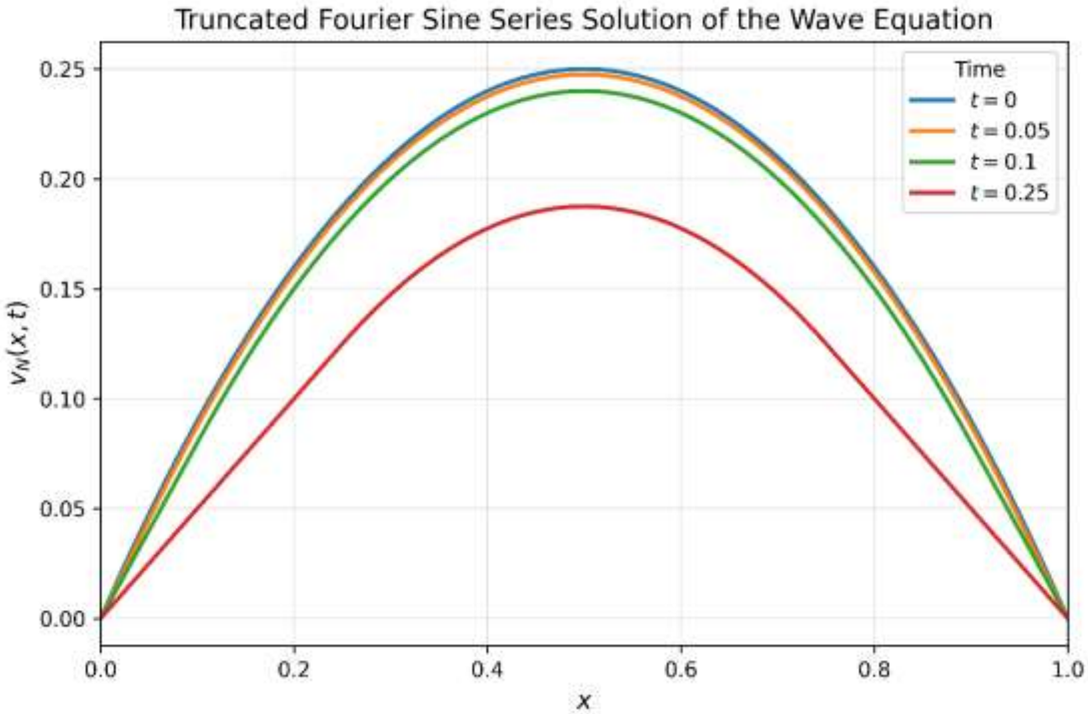

Figure 2. Truncated Fourier sine series solution $v_N(x,t)$of the wave equation for $f(x) = x(1-x)$, $g(x) = 0$, $L = 1$, $c = 1$, and $N = 50$, at selected times.

11. Discussion of Results

The analytical and graphical findings validate that the one-dimensional heat and wave equations share a common spatial structure, yet exhibit fundamentally distinct time evolution. When subjected to the same homogeneous Dirichlet boundary conditions, both equations yield the sine eigenfunctions.

$$\sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots .$$

This suggests that the boundary conditions dictate the spatial modes in both models.
Consequently, the primary difference between the two equations does not stem from the spatial component, but rather from the temporal component of the separated solution.For the heat equation, the factor

$$e^{-\kappa\lambda_n t}$$

Each Fourier mode experiences exponential decay. As λn increases with $n2$, high-frequency modes diminish more quickly than their low-frequency counterparts. This phenomenon elucidates why the heat equation smooths the initial data and propels the solution towards the zero equilibrium state when subjected to homogeneous Dirichlet boundary conditions. The numerical example using $f(x)$=x(1−x) reinforces this conclusion, as the heat profiles depicted in the plot show a reduction in amplitude over time.

For the wave equation, the temporal factors

$$\cos\,(c\sqrt{\lambda_n}t)\text{and}\sin\,(c\sqrt{\lambda_n}t)$$

produce oscillatory motion rather than decay. The frequencies increase with n, so higher modes oscillate more rapidly. However, unlike the heat equation, these modes are not suppressed by an exponential damping factor. As a result, the wave equation preserves oscillatory behavior in the ideal undamped case. The graphical illustration with $g(x) = 0$demonstrates this behavior, since the solution continues to oscillate instead of approaching equilibrium.

The analysis further indicates that the sequence of the time derivative is crucial in determining the qualitative characteristics of the solution. The first-order time derivative present in the heat equation results in a dissipative evolution, while the second-order time derivative found in the wave equation results in a conservative oscillatory evolution. Therefore, despite both equations possessing the same eigenfunctions under the specified boundary conditions, their solutions exhibit notable differences in long-term behavior, handling of high-frequency modes, and physical interpretation.

These findings reinforce the primary aim of the research: to demonstrate that the distinction between diffusion and wave propagation can be mathematically elucidated through the temporal equations derived from the separation of variables. The heat equation signifies smoothing and decay, whereas the wave equation signifies vibration and enduring modal motion.

12. Conclusion

This research conducted a comparative analytical examination of the one-dimensional heat and wave equations through the method of separation of variables.

Both equations were analyzed within a finite interval while adhering to homogeneous Dirichlet boundary conditions. This shared framework resulted in an identical spatial eigenvalue problem and a corresponding family of orthogonal sine eigenfunctions.

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1,2,3, \dots .$$

The analysis showed that the essential difference between the two models lies in the temporal component of the separated solutions. For the heat equation, each Fourier mode is multiplied by the exponentially decaying factor

$$e^{-\kappa\lambda_n t},$$

which produces diffusion, smoothing, damping of high-frequency modes, and convergence toward equilibrium. For the wave equation, the corresponding temporal factors are

$$\cos\left(c\sqrt{\lambda_n}t\right)\text{and}\sin\left(c\sqrt{\lambda_n}t\right),$$

which generate enduring oscillatory behavior in the absence of damping.

The comparative analysis additionally revealed that the order of the time derivative is crucial in determining the qualitative behavior of each equation. The first-order time derivative in the heat equation results in dissipative evolution, while the second-order time derivative in the wave equation results in conservative oscillatory motion. Consequently, although both equations possess the same spatial eigenfunctions under the specified boundary conditions, their long-term behavior and handling of high-frequency modes are fundamentally distinct.

The numerical examples based on truncated Fourier sine series corroborated the theoretical findings. The solution to the heat equation diminished in amplitude and approached equilibrium, whereas the solution to the wave equation sustained oscillatory motion. Thus, the study validates that the difference between diffusion and wave propagation is mathematically represented in the temporal component of the separated solutions. This offers a clear comparison between parabolic and hyperbolic partial differential equations within a cohesive analytical framework.k.

References:

[1] Azouz, A., Fawzi, M., Mohammed, I., Hamed, O., Maher, A., & Baddi, M. (2026). Influence of Electrolyte Chemistry and Electrode Material on Hydrogen Production

Performance in Alkaline Water Electrolysis. *Al-Farooq Journal of Sciences*, *2*(2), 49-66.

[2]Dalla, L. O. B., Essgaer, M., Jetlawei, S. S., EL-sseid, M., Alsharif, A., & Agila, A. A. A. (2026). Local Precision and Global Harmony: A Comparative Literature Review (LR) Framework for Taylor and Fourier Series in Engineering Modeling. *Al-Farooq Journal of Sciences*, *2*(1), 275-304.

[3] D. Borthwick, *Introduction to Partial Differential Equations*. Cham, Switzerland: Springer, 2016. doi: 10.1007/978-3-319-48936-0.

[4] R. Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, 5th ed. Boston, MA: Pearson, 2013.

[5] A. S. Fokas and E. A. Spence, "Synthesis, as opposed to separation, of variables," *SIAM Review*, vol. 54, no. 2, pp. 291–324, 2012. doi: 10.1137/100809647.

[4] F. Olivar-Romero and O. Rosas-Ortiz, "Transition from the wave equation to either the heat or the transport equations through fractional differential expressions," *Symmetry*, vol. 10, no. 10, Art. no. 524, 2018. doi: 10.3390/sym10100524.

[6] L. Grafakos, *Classical Fourier Analysis*, 3rd ed. New York, NY: Springer, 2014. doi: 10.1007/978-1-4939-1194-3.

[7] V. Serov, *Fourier Series, Fourier Transform and Their Applications to Mathematical Physics*. Cham, Switzerland: Springer, 2017. doi: 10.1007/978-3-319-65262-7.

[8] R. P. Agarwal and D. O'Regan, *Ordinary and Partial Differential Equations: With Special Functions, Fourier Series, and Boundary Value Problems*. New York, NY: Springer, 2009. doi: 10.1007/978-0-387-79146-3.

[9] A. Zettl, *Sturm–Liouville Theory*. Providence, RI: American Mathematical Society, 2005.

[10] J. R. Cannon, *The One-Dimensional Heat Equation*. Cambridge, UK: Cambridge University Press, 1984. doi: 10.1017/CBO9781139086967.

[11] M. A. Pinsky, *Partial Differential Equations and Boundary-Value Problems with Applications*, 3rd ed. Providence, RI: American Mathematical Society, 2011.

[12] L. C. Evans, *Partial Differential Equations*, 2nd ed. Providence, RI: American Mathematical Society, 2010.

[13] W. A. Strauss, *Partial Differential Equations: An Introduction*, 2nd ed. Hoboken, NJ: Wiley, 2007.

[14] M. E. Taylor, *Partial Differential Equations I: Basic Theory*, 2nd ed. New York, NY: Springer, 2011. doi: 10.1007/978-1-4419-7055-8.

[15] F. John, *Partial Differential Equations*. New York, NY: Springer, 1978. doi: 10.1007/978-1-4684-0059-5.

[16] J. S. Hesthaven, S. Gottlieb, and D. Gottlieb, *Spectral Methods for Time-Dependent Problems*. Cambridge, UK: Cambridge University Press, 2007. doi: 10.1017/CBO9780511618352.

[17] J. Shen, T. Tang, and L.-L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*. Berlin, Germany: Springer, 2011. doi: 10.1007/978-3-540-71041-7.

[18] L. N. Trefethen, *Spectral Methods in MATLAB*. Philadelphia, PA: SIAM, 2000. doi: 10.1137/1.9780898719598.

[19] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, 2nd ed. Mineola, NY: Dover Publications, 2001.

[20] R. Shamshiri and W. I. Wan Ismail, "A comprehensive comparison between wave propagation and heat distribution via analytical solutions and computer simulations," *Research Journal of Applied Sciences, Engineering and Technology*, vol. 7, no. 1, pp. 111–122, 2014. doi: 10.19026/rjaset.7.228.