Comprehensive Journal of Science
Volume (11), Issue (41), (JUNE, 2026)
ISSN: 3014-6266

مجلة العلوم الشاملة
المجلد (11) العدد 41 يونيو 2026
ردمد: 3014-6266

Some indirect iterative methods for solving Poisson's incomplete differential equation

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تاريخ الاستلام:2026/4/03 –تاريخ المراجعة:2026/05/04– تاريخ القبول: 2026/05/16– تاريخ النشر04/ 06/ 2026

Abstract:
Second-order incomplete partial differential equations were studied using numerical methods. For this type of incomplete partial differential equation, we use the rule of fives. To transform the incomplete Poisson equation into a differential equation, we use the Jacobi, Gauss-Seidel, and Lippmann iteration methods.

Keywords: Poisson equation – Iteration methods (Gauss-Seidel, Lippmann, Jacobi).

Introduction: When solving decreasing differential equations, we obtain a very large number of algebraic equations, sometimes reaching tens of thousands, which are difficult to solve. Finding all five points at once improves accuracy using any iterative method for solving the incomplete Poisson differential equation, as the Poisson equation states a five-point difference relationship.

Solution of Poisson's Equation

Its method of solution is similar to that of that Laplace equation. Here the standard five- point formula for $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$

Takes the form

$$T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = h^2 f(ih, jh)$$

By applying

$$T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = h^2 f(ih, jh)$$

At eatch interior mesh point, we arrive at linear equtations in the nodal values $T_{i,j}$ These equations can be solved by the Gauss – Seidal method.

Jacobi's method (for the incomplete Poisson equation)

We will demonstrate this method by addressing the incomplete Poisson equation.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y) \qquad [1]$$

Let the solution region be the rectangular region h of dimensions $mS\ , nS$. Let the solution ω be known on the boundaries of region h. By dividing this region into a grid of squares with steps S, where the points of the grid are defined by the figure

$xi = is \quad , i = 0,1,2, \dots m$
$yi = js \quad , j = 0,1,2, \dots n$

Then we take the previous differential equation in the following differential form

$$T_{i-1,j} + T_{i,j-1} - 4T_{i,j} + T_{i+1,j} + T_{i,j+1} + S^2 F_{ij} = 0$$

Which is given in the form

$$T_{i,j} = \frac{1}{4}\left(T_{i-1,j} + T_{i,j-1} + T_{i+1,j} + T_{i,j+1} + S^2 F_{i,j}\right) \qquad [4]$$

Jacobi's method for this difference equation is

$$T_{i,j}^{(n+1)} = \frac{1}{4}\left(T_{i-1,j}^{(n)} + T_{i,j-1}^{(n)} + T_{i+1,j}^{(n)} + T_{i,j+1}^{(n)} + S^2 F_{i,j}\right) \quad [2]$$

For points within the network, and for points on the borders (circumferring area h), we have

$$T_{i,j}^{(n+1)} = c_{ij} \ ; \ i = o\ , m \ ; \ j = o\ , n$$

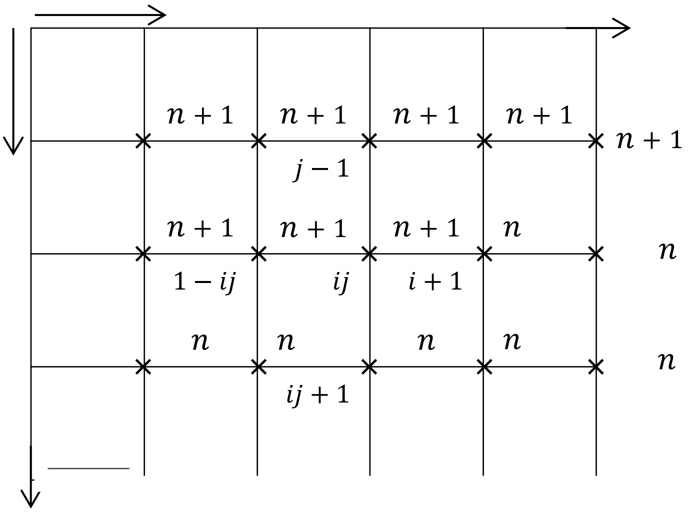
Its acceleration is proportional to the following Jacobi method.

$$m = P(J) = \frac{1}{2}\left(\cos\frac{\pi}{m} + \cos\frac{\pi}{n}\right)$$

The Gauss-Seidel method (for the incomplete Poisson differential treatment)

This method differs from the previous one.

If we assume that the frequency values of the solution in step $(n+1)$ calculated for $J = 1,2,\ldots j^{-1}$ and also according to the models have been calculated up to the region $(i-1\,, j)$, then the value of the solution at the point $(i\,, j)$ and in step $(n+1)$ is given according to the following form:



The third relationship

$$T_{i,j}^{(n+1)} = \frac{1}{4}\left(T_{i-1,j}^{(n+1)} + T_{i,j-1}^{(n+1)} + T_{i+1,j}^{(n)} + T_{i,j+1}^{(n)} + S^2F_{i,j}\right)$$

This relationship is often called the "Liebmann method of external interpolation" according to the Gauss-Seidel scheme of Poisson's equation. [4]

The convergence of this method occurs when $K_{ij} \to 0$ where:

$$K_{i,j} = T_{i-1,j}^{(n+1)} + T_{i,j-1}^{(n+1)} + T_{i+1,j}^{(n)} + T_{i,j+1}^{(n)} - 4T_{ij}^{(n)} + S^2F_{i,j}$$

Where $4T_{ij}^{(n+1)}$ in the previous difference equation

$T_{i,j}^{(n+1)} = \frac{1}{4}\left(T_{i-1,j}^{(n+1)} + T_{i,j-1}^{(n+1)} + T_{i+1,j}^{(n)} + T_{i,j+1}^{(n)} + S^2F_{i,j}\right)$ [3]

It is an improved value of $T_{i,j}^{(n)}$

The error in the step-repetition $(n+1)$ is given by the expression $\left|e^{(n+1)}\right| = \left|e^{(n)}\right|$ where e is the error at any point in the grid between the exact solution and the approximate solution obtained from solving the previous difference equations, and m is the modulus of the repetition matrix. In this method, m is given by the relation

$$ln = ln(G) = ln(J)^2 = \frac{1}{4}\left(cos\frac{\pi}{m} + cos\frac{\pi}{n}\right)^2$$

The Gauss-Seidel method is twice as fast as the Jacobi method, and this convergence is related to $m\,, n$. The smaller the grid step – i.e., the larger $m\,, n$– the smaller the convergence ratio. [4]

- **Lippmann's method of external interpolation (for the elliptical Poisson differential equation)**

The differential equation in the previous Gauss-Seidel method:

$$T_{i,j}^{(n+1)} = \frac{1}{4}\left(T_{i-1,j}^{(n+1)} + T_{i,j-1}^{(n+1)} + T_{i+1,j}^{(n)} + T_{i,j+1}^{(n)} - 4T_{ij}^{(n)} + S^2F_{i,j}\right)$$

It is written in the following format:

$$T_{i,j}^{(n+1)} = T_{i,j}^{(n)}\frac{1}{4}\left(T_{i-1,j}^{(n+1)} + T_{i-1,j}^{(n)} + T_{i,j-1}^{(n+1)} + T_{i,j+1}^{(n)} - 4T_{i,j}^{(n)} + S^2F_{i,j}\right)$$

$$= T_{i,j}^{(n)} + \frac{1}{4}K_{ij}$$

Where $\frac{1}{4}K_{i,j}$ is the change in the value of T_{ij} during one iteration step in the Gaussian-Seidel method, and the iteration from the origin of the internal points of the network is then given by the relation:

$$T_{ij}^{(n+1)} = T_{i,j}^{(n)} + \frac{1}{4}GK_{i,j}$$

$$= T_{i,j}^{(n)} + \lambda\left(T_{i-1,j}^{(n+1)} + T_{i-1,j}^{(n)} + T_{i-1,j}^{(n+1)} + T_{i,j+1}^{(n)} - 4T_{i,j}^{(n)} + S^2F_{ij}\right)$$

Whereas $\frac{1}{4} \le \lambda \le \frac{1}{2}$, $\lambda = \frac{1}{4}\theta$

The last relation can be written in terms of θ as follows:

$$T_{i,j}^{(n+1)} = \theta\left\{\frac{1}{4}\left(T_{i-1,j}^{(n+1)} + T_{i+1,j}^{(n)} + T_{i,j-1}^{(n+1)} + T_{i,j-1}^{(n)} + S^2F_{i,j}\right)\right\} + (1-\theta)T_{i,j}^{(n)}$$

This repetition, according to this method, is a linear combination with the third Gaussian-Seidel relation.

$$T_{i,j}^{(n+1)} = \frac{1}{4}\left(T_{i-1,j}^{(n+1)} + T_{i,j-1}^{(n+1)} + T_{i+1,j}^{(n)} + T_{i,j+1}^{(n)} + S^2F_{i,j}\right)$$

With the repetition $T_{i,j}^{(n)}$ of step n.

EXAMPLE :-

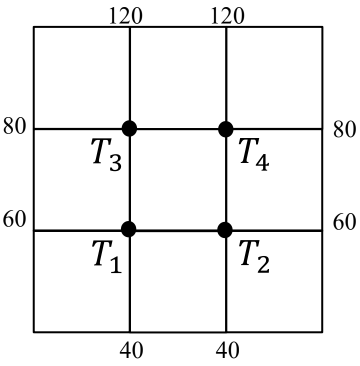
Solve the Piosson equation

$\nabla^2 T = -10, h = 1$

In th domain by

a- Jacobi's method.
b- Gauss-Seidal method

Solution :



The standard five- point formula for the given equation is

$$T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = h^2 f(ih, jh)$$

$T_2 + 60 + T_3 + 40 - 4T_1 = -10$
$60 + T_1 + T_4 + 40 - 4T_2 = -20$
$T_4 + 80 + 120 + T_1 - 4T_3 = -10$
$80 + T_3 + 120 + T_2 - 4T_4 = -20$

$$T_1^{(n)} = \frac{1}{4}[T_2 + T_3 + 110]$$

$$T_2^{(n)} = \frac{1}{4}[T_1 + T_4 + 120]$$

$$T_3^{(n)} = \frac{1}{4}[T_1 + T_4 + 210]$$

$$T_4^{(n)} = \frac{1}{4}[T_2 + T_3 + 220]$$

We carry out the successive iterations, using Jacobi's formulas

$$T_1^{(0)} = 0 \ , \ T_2^{(0)} = 0 \ , \ T_3^{(0)} = 0 \ , \ T_4^{(0)} = 0$$

$$T_1^{(1)} = \frac{1}{4}\left[T_2^{(0)} + T_3^{(0)} + 110\right] = \frac{1}{4}[0 + 0 + 110] = \mathbf{27.5}$$
$$T_2^{(1)} = \frac{1}{4}\left[T_1^{(0)} + T_4^{(0)} + 120\right] = \frac{1}{4}[0 + 0 + 120] = \mathbf{30}$$
$$T_3^{(1)} = \frac{1}{4}\left[T_1^{(0)} + T_4^{(0)} + 210\right] = \frac{1}{4}[0 + 0 + 210] = \mathbf{52.5}$$
$$T_4^{(1)} = \frac{1}{4}[T_2 + T_3 + 220] = \frac{1}{4}[0 + 0 + 220] = \mathbf{55}$$
$$\boldsymbol{T_1^{(1)} = 27.5 \ , \ T_2^{(1)} = 30 \ , \ T_3^{(1)} = 52.5 \ , \ T_4^{(1)} = 55}$$

$$T_1^{(2)} = \frac{1}{4}[30 + 52.5 + 110] = \mathbf{48.125}$$
$$T_2^{(2)} = \frac{1}{4}[27.5 + 55 + 120] = \mathbf{50.625}$$
$$T_3^{(2)} = \frac{1}{4}[27.5 + 55 + 210] = \mathbf{73.125}$$
$$T_4^{(2)} = \frac{1}{4}[30 + 52.5 + 270] = \mathbf{75.625}$$
$$\boldsymbol{T_1^{(2)} = 48.125, \ T_2^{(2)} = 50.625, \ T_3^{(2)} = 73.125, \ T_4^{(2)} = 75.625}$$

$$T_1^{(3)} = \frac{1}{4}[50.625 + 73.125 + 110] = \mathbf{58.4375}$$
$$T_2^{(3)} = \frac{1}{4}[48.125 + 75.625 + 120] = \mathbf{60.9375}$$
$$T_3^{(3)} = \frac{1}{4}[48.125 + 75.625 + 210] = \mathbf{83.4375}$$
$$T_4^{(3)} = \frac{1}{4}[50.625 + 73.125 + 220] = \mathbf{85.9375}$$
$$\boldsymbol{T_1^{(3)} = 58.4375, \ T_2^{(3)} = 60.9375, \ T_3^{(3)} = 83.4375, \ T_4^{(3)} = 85.9375}$$

$$T_1^{(4)} = \frac{1}{4}[60.9375 + 83.4375 + 110] = \mathbf{63.603125}$$
$$T_2^{(4)} = \frac{1}{4}[58.4375 + 85.9375 + 120] = \mathbf{66.09375}$$
$$T_3^{(4)} = \frac{1}{4}[58.4375 + 85.9375 + 210] = \mathbf{88.46875}$$
$$T_4^{(4)} = \frac{1}{4}[60.9375 + 83.4375 + 220] = \mathbf{91.09375}$$
$$\boldsymbol{T_1^{(4)} = 63.603125, \ T_2^{(4)} = 66.09375, \ T_3^{(4)} = 88.46875, \ T_4^{(4)} = 91.09375}$$

$$T_1^{(5)} = \frac{1}{4}[66.09375 + 88.46875 + 110] = \mathbf{66.140625}$$
$$T_2^{(5)} = \frac{1}{4}[63.603125 + 91.09375 + 120] = \mathbf{68.67421875}$$
$$T_3^{(5)} = \frac{1}{4}[63.603125 + 91.09375 + 210] = \mathbf{91.17471875}$$
$$T_4^{(5)} = \frac{1}{4}[66.09375 + 88.46875 + 220] = \mathbf{93.640625}$$

We cqrry out the successive iteration, using Gauss-Seidal formals

$T_2^{(0)} = 0,\ T_3^{(0)} = 0,\ T_4^{(0)} = 0,$

$T_1^{(n+1)} = \frac{1}{4}\left[T_2^{(0)} + T_3^{(0)} + 110\right]$

$T_2^{(n+1)} = \frac{1}{4}\left[T_1^{(n+1)} + T_4^{(0)} + 120\right]$

$T_3^{(n+1)} = \frac{1}{4}\left[T_1^{(n+1)} + T_4^{(n+1)} + 210\right]$

$T_4^{(n+1)} = \frac{1}{4}\left[T_2^{(n+1)} + T_3^{(n+1)} + 220\right]$

$T_1^{(1)} = \frac{1}{4}[0 + 0 + 110] = 27.5$

$T_2^{(1)} = \frac{1}{4}[27.5 + 0 + 120] = 36.875$

$T_3^{(1)} = \frac{1}{4}[27.5 + 0 + 210] = 59.375$

$T_4^{(1)} = \frac{1}{4}[36.875 + 59.375 + 220] = 79.0625$

$T_1^{(2)} = \frac{1}{4}[36.875 + 59.375 + 110] = 51.5625$

$T_2^{(2)} = \frac{1}{4}[51.5625 + 79.0625 + 120] = 62.65625$

$T_3^{(2)} = \frac{1}{4}[51.5625 + 79.0625 + 210] = 85.15625$

$T_4^{(2)} = \frac{1}{4}[62.65625 + 85.15625 + 220] = 91.953125$

$T_1^{(3)} = \frac{1}{4}[62.65625 + 85.15625 + 110] = 64.453125$

$T_2^{(3)} = \frac{1}{4}[64.453125 + 91.953125 + 120] = 69.1015625$

$T_3^{(3)} = \frac{1}{4}[64.453125 + 91.953125 + 210] = 91.6015625$

$T_4^{(3)} = \frac{1}{4}[69.1015625 + 91.6015625 + 220] = 95.17578125$

$T_1^{(4)} = \frac{1}{4}[69.015625 + 91.6015625 + 110] = 67.67578125$

$T_2^{(4)} = \frac{1}{4}[67.67578125 + 95.17578125 + 120] = 70.71289063$

$T_3^{(4)} = \frac{1}{4}[67.67578125 + 95.17578125 + 210] = 93.21289063$

$T_4^{(4)} = \frac{1}{4}[70.71289063 + 93.2189063 + 220] = 95.98294923$

$T_1^{(5)} = \frac{1}{4}[70.7189063 + 93.21289063 + 110] = 68.48144532$

$T_2^{(5)} = \frac{1}{4}[68.48144532 + 95.98294923 + 120] = 71.11609864$

$$T_3^{(5)} = \frac{1}{4}[68.48144532 + 95.98294923 + 210] = 93.61609864$$

$$T_4^{(5)} = \frac{1}{4}[71.11609864 + 93.61609864 + 220] = 96.18304932$$

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